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CLASSICAL AND QUANTUM ORTHOGONAL POLYNOMIALS IN ONE VARIABLE

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Classical and Quantum Orthogonal Polynomials in One Variable

This is first modern treatment of orthogonal polynomials from the viewpoint of special functions. The coverage is encyclopedic, including classical topics such as Jacobi, Hermite, Laguerre, Hahn, Charlier and Meixner polynomials as well as those, e.g. Askey–Wilson and Al-Salam–Chihara polynomial systems, discovered over the last 50 years: multiple orthogonal polynomials are dicussed for the first time in book form. Many modern applications of the subject are dealt with, including birth and death processes, integrable systems, combinatorics, and physical models. A chapter on open research problems and conjectures is designed to stimulate further research on the subject.

Exercises of varying degrees of difficulty are included to help the graduate student and the newcomer. A comprehensive bibliography rounds off the work, which will be valued as an authoritative reference and for graduate teaching, in which role it has already been successfully class-tested.

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With two chapters by

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Foreword

There are a number of ways of studying orthogonal polynomials. Gabor Szegő's book "Orthogonal Polynomials" (Szegő, 1975) had two main topics. Most of this book dealt with polynomials which are orthogonal on the real line, with a chapter on polynomials orthogonal on the unit circle and a short chapter on polynomials orthogonal on more general curves. About two-thirds of Szegő's book deals with the classical orthogonal polynomials of Jacobi, Laguerre and Hermite, which are orthogonal with respect to the beta, gamma and normal distributions, respectively. The rest deals with more general sets of orthogonal polynomials, some general theory, and some asymptotics.

Barry Simon has recently written a very long book on polynomials orthogonal on the unit circle, (Simon, 2004). His book has very little on explicit examples, so its connection with Szegő's book is mainly in the general theory, which has been developed much more deeply than it had been in 1938 when Szegő's book appeared.

The present book, by Mourad Ismail, complements Szegő's book in a different way. It primarily deals with specific sets of orthogonal polynomials. These include the classical polynomials mentioned above and many others. The classical polynomials of Jacobi, Laguerre and Hermite satisfy second-order linear homogeneous differential equations of the form

$$a(x)y''(x) + b(x)y'(x) + \lambda_n y(x) = 0$$

where $a(x)$ and $b(x)$ are polynomials of degrees 2 and 1, respectively, which are independent of n, and λ_n is independent of x. They have many other properties in common. One is that the derivative of $p_n(x)$ is a constant times $q_{n-1}(x)$ where $\{p_n(x)\}$ is in one of these classes of polynomials and $\{q_n(x)\}$ is also. These are the only sets of orthogonal polynomials with the property that their derivatives are also orthogonal.

Many of the classes of polynomials studied in this book have a similar nature, but with the derivative replaced by another operator. The first operator which was used is

$$\Delta f(x) = f(x+1) - f(x),$$

a standard form of a difference operator. Later, a q-difference operator was used

$$D_q f(x) = [f(qx) - f(x)]/[qx - x].$$

Still later, two divided difference operators were introduced. The orthogonal polynomials which arise when the q-divided difference operator is used contain a set of polynomials introduced by L. J. Rogers in a remarkable series of papers which appeared in the 1890s. One of these sets of polynomials was used to derive what we now call the Rogers–Ramanujan identities. However, the orthogonality of Rogers's polynomials had to wait decades before it was found. Other early work which leads to polynomials in the class of these generalized classical orthogonal polynomials was done by Chebyshev, Markov and Stieltjes.

To give an idea about the similarities and differences of the classical polynomials and some of the extensions, consider a set of polynomials called ultraspherical or Gegenbauer polynomials, and the extension Rogers found. Any set of polynomials which is orthogonal with respect to a positive measure on the real line satisfies a three term recurrence relation which can be written in a number of equivalent ways. The ultraspherical polynomials $C_n^\nu(x)$ are orthogonal on $(-1,1)$ with respect to $\left(1-x^2\right)^{\nu-1/2}$. Their three-term recurrence relation is

$$2\,(n+\nu)\,xC_n^\nu(x) = (n+1)\,C_{n+1}^\nu(x) + (n+2\nu-1)\,C_{n-1}^\nu(x)$$

The three-term recurrence relation for the continuous q-ultraspherical polynomials of Rogers satisfy a similar recurrence relation with every $(n+a)$ replaced by $1-q^{n+a}$. That is a natural substitution to make, and when the recurrence relation is divided by $1-q$, letting q approach 1 gives the ultraspherical polynomials in the limit.

Both of these sets of polynomials have nice generating functions. For the ultraspherical polynials one nice generating function is

$$\left(1-2xr+r^2\right)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(x)\,r^n$$

The extension of this does not seem quite as nice, but when the substitution $x = \cos\theta$ is used on both, they become similar enough for one to guess what the left-hand side should be. Before the substitution it is

$$\prod_{n=0}^{\infty} \frac{\left(1-2xq^{\nu+n}r+q^{2\nu+2n}r^2\right)}{\left(1-2xq^n r+q^{2n}r^2\right)} = \sum_{n=0}^{\infty} C_n(x;q^\nu\,|\,q)\,r^n.$$

The weight function is a completely different story. To see this, it is sufficient to state it:

$$w\,(x,q^\nu) = \left(1-x^2\right)^{-1/2} \prod_{n=0}^{\infty} \frac{\left(1-\left(2x^2-1\right)q^n+q^{2n}\right)}{\left(1-\left(2x^2-1\right)q^{n+\nu}+q^{2n+2\nu}\right)}.$$

These polynomials of Rogers were rediscovered about 1940 by two mathematicians, (Feldheim, 1941b) and (Lanzewizky, 1941). Enough had been learned about orthogonal polynomials by then for them to know they had sets of orthogonal polynomials, but neither could find the orthogonality relation. One of these two mathematicians, E. Feldheim, lamented that he was unable to find the orthogonality relation. Stieltjes and Markov had found theorems which would have allowed Felheim to work out the orthogonality relation, but there was a war going on when Feldheim did his work and he was unaware of this old work of Stieltjes and Markov. The limiting case when

$\nu \to \infty$ gives what are called the continuous q-Hermite polynomials. It was these polynomials which Rogers used to derive the Rogers-Ramanujan identities.

Surprisingly, these polynomials have recently come up in a very attractive problem in probability theory which has no q in the statement of the problem. See Bryc (Bryc, 2001) for this work.

Stieltjes solved a minimum problem which can be considered as coming from one dimensional electrostatics, and in the process found the discriminant for Jacobi polynomials. The second-order differential equation they satisfy played an essential role. When I started to study special functions and orthogonal polynomials, it seemed that the only orthogonal polynomials which satisfied differential equations nice enough to be useful were Jacobi, Laguerre and Hermite. For a few classes of orthogonal polynomials nice enough differential equations existed, but they were not well known. Now, thanks mainly to a conjecture of G. Freud which he proved in two very special cases, and work by quite a few people including Nevai and some of his students, we know that nice enough differential equations exist for polynomials orthogonal with respect to $\exp(-v(x))$ when $v(x)$ is smooth enough. The work of Stieltjes can be partly extended to this much wider class of orthogonal polynomials. Some of this is done in Chapter 3.

Chapter 4 deals with the classical polynomials. For Hermite polynomials there is an explicit expression for the analogue of the Poisson kernel for Fourier series which was found by Mehler in the 19th century. An important multivariable extension of this formula found independently by Kibble and Slepian is in Chapter 4. Chapter 5 contains some information about the Pollaczek polynomials on the unit interval. Their recurrence relation is a slight variant of the one for ultraspherical polynomials listed above. The weight function is drastically different, having infinitely many point masses outside the interval where the absolutely continuous part is supported or vanishing very rapidly at one or both of the end points of the interval supporting the absolutely continuous part of the orthogonality measure.

Chapter 6 deals with extensions of the classical orthogonal polynomials whose weight function is discrete. Here the classical discriminant seemingly cannot be found in a useful form, but a variant of it has been computed for the Hahn polynomials. This extends the result of Stieltjes on the discriminant for Jacobi polynomails. Hahn polynomials extend Jacobi polynomials and are orthogonal with respect to the hypergeometric distribution. Transformations of them occur in the quantum theory of angular momentum and they and their duals occur in some settings of coding theory.

The polynomials considered in the first 10 chapters which have explicit formulas are given as generalized hypergeometric series. These are series whose term ratio is a rational function of n. In Chapters 11 to 19 a different setting occurs, that of basic hypergeometric series. These are series whose term ratio is a rational function of q^n.

In the 19th century Markov and Stieltjes found examples of orthogonal polynomials which can be written as basic hypergeometric series and found an explicit orthogonality relation. As mentioned earlier, Rogers also found some polynomials which are orthogonal and can be given as basic hypergeometric series, but he was unaware they were orthogonal. A few other examples were found before Wolfgang Hahn wrote a major paper, (Hahn, 1949b) in which he found basic hypergeometric extensions of the classical polynomials and the discrete ones up to the Hahn polynomial level. There is one level higher than this where orthogonal polynomials exist which have properties very similar to many of those known for the classical orthogonal polynomials. In particular, they satisfy a second-order divided q-difference equation and this divided q-difference operator applied to them gives another set of orthogonal polynomials. When this was first published, the polynomials were treated directly without much motivation. Here simpler cases are done first and then a boot-strap argument allows one to obtain more general polynomials, eventually working up to the most general classical type sets of orthogonal polynomials.

The most general of these polynomials has four free parameters in addition to the q of basic hypergeometric series. When three of the parameters are held fixed and the fourth is allowed to vary, the coefficients which occur when one is expanded in terms of the other are given as products. The resulting identity contains a very important transformation formula between a balanced ${}_4\phi_3$ and a very-well-poised ${}_8\phi_7$ which Watson found in the 1920s as the master identity which contains the Rogers-Ramanujan identities as special cases and many other important formulas. There are many ways to look at this identity of Watson, and some of these ways lead to interesting extensions. When three of the four parameters are shifted and this connection problem is solved, the coefficients are single sums rather than the double sums which one expects. At present we do not know what this implies, but surprising results are usually important, even if it takes a few decades to learn what they imply.

The fact that there are no more classical type polynomials beyond those mentioned in the last paragraph follows from a theorem of Leonard (Leonard, 1982). This theorem has been put into a very attractive setting by Terwilliger, some of whose work has been summarized in Chapter 20. However, that is not the end since there are biorthogonal rational functions which have recently been discovered. Some of this work is contained in Chapter 18. There is even one higher level than basic hypergeometric functions, elliptic hypergeometric functions. Gasper and Rahman have included a chapter on them in (Gasper & Rahman, 2004).

Chapters 22 and 23 were written by Walter Van Assche. The first is on the Riemann-Hilbert method of studying orthogonal polynomials. This is a very powerful method for deriving asymptotics of wide classes of orthogonal polynomials. The second chapter is on multiple orthogonal polynomials. These are polynomials in one variable which are orthogonal with respect to r different measures. The basic ideas go back to the 19th century, but except for isolated work which seems to start with Angelesco in 1919, it has only been in the last 20 or so years that significant work has been done on them.

There are other important results in this book. One which surprised me very much is the q-version of Airy functions, at least as the two appear in asymptotics. See, for example, Theorem 21.7.3.

When I started to work on orthogonal polynomials and special functions, I was told by a number of people that the subject was out-of-date, and some even said dead. They were wrong. It is alive and well. The one variable theory is far from finished, and the multivariable theory has grown past its infancy but not enough for us to be able to predict what it will look like in 2100.

Madison, WI Richard A. Askey
April 2005

Preface

I first came across the subject of orthogonal polynomials when I was a student at Cairo University in 1964. It was part of a senior-level course on special functions taught by the late Professor Foad M. Ragab. The instructor used his own notes, which were very similar in spirit to the way Rainville treated the subject. I enjoyed Ragab's lectures and, when I started graduate school in 1968 at the Univerity of Alberta, I was fortunate to work with Waleed Al-Salam on special functions and q-series. Jerry Fields taught me asymptotics and was very generous with his time and ideas. In the late 1960s, courses in special functions were a rarity at North American universities and have been replaced by Bourbaki-type mathematics courses. In the early 1970s, Richard Askey emerged as the leader in the area of special functions and orthogonal polynomials, and the reader of this book will see the enormous impact he made on the subject of orthogonal polynomials. At the same time, George Andrews was promoting q-series and their applications to number theory and combinatorics. So when Andrews and Askey joined forces in the mid-1970s, their combined expertise advanced the subject in leaps and bounds. I was very fortunate to have been part of this group and to participate in these developments. My generation of special functions / orthogonal polynomials people owes Andrews and Askey a great deal for their ideas which fueled the subject for a while, for the leadership role they played, and for taking great care of young people.

This book project started in the early 1990s as lecture notes on q-orthogonal polynomials with the goal of presenting the theory of the Askey–Wilson polynomials in a way suitable for use in the classroom. I taught several courses on orthogonal polynomials at the University of South Florida from these notes, which evolved with time. I later realized that it would be better to write a comprehensive book covering all known systems of orthogonal polynomials in one variable. I have attempted to include as many applications as possible. For example, I included treatments of the Toda lattice and birth and death processes. Applications of connection relations for q-polynomials to the evaluation of integrals and the Rogers–Ramanujan identities are also included. To the best of my knowledge, my treatment of associated orthogonal polynomials is a first in book form. I tried to include all systems of orthogonal polynomials but, in order to get the book out in a timely fashion, I had to make some compromises. I realized that the chapters on Riemann–Hilbert problems and multiple orthogonal polynomials should be written by an expert on the subject, and

Walter Van Assche kindly agreed to write this material. He wrote Chapters 22 and 23, except for §22.8. Due to the previously mentioned time constraints, I was unable to treat some important topics. For example, I covered neither the theories of matrix orthogonal polynomials developed by Antonio Durán, Yuan Xu and their collaborators, nor the recent interesting explicit systems of Grünbaum and Tirao and of Durán and Grünbaum. I hope to do so if the book has a second edition. Regrettably, neither the Sobolov orthogonal polynomials nor the elliptic biorthogonal rational functions are treated.

Szegő's book on orthogonal polynomials inspired generations of mathematicians. The character of this volume is very different from Szegő's book. We are mostly concerned with the special functions aspects of orthogonal polynomials, together with some general properties of orthogonal polynomial systems. We tried to minimize the possible overlap with Szegő's book. For example, we did not treat the refined bounds on zeros of Jacobi, Hermite and Laguerre polynomials derived in (Szegő, 1975) using Sturmian arguments. Although I tried to cover a broad area of the subject matter, the choice of the material is influenced by the author's taste and personal bias.

Dennis Stanton has used parts of this book in a graduate-level course at the University of Minnesota and kindly supplied some of the exercises. His careful reading of the book manuscript and numerous corrections and suggestions are greatly appreciated. Thanks also to Richard Askey and Erik Koelink for reading the manuscript and providing a lengthy list of corrections and additional information. I am grateful to Paul Terwilliger for his extensive comments on §20.3.

I hope this book will be useful to students and researchers alike. It has a collection of open research problems in Chapter 24 whose goal is to challenge the reader's curiosity. These problems have varying degrees of difficulty, and I hope they will stimulate further research in this area.

Many people contributed to this book directly or indirectly. I thank the graduate students and former graduate students at the University of South Florida who took orthogonal polynomials and special functions classes from me and corrected misprints. In particular, I thank Plamen Simeonov, Jacob Christiansen, and Jemal Gishe. Mahmoud Annaby and Zeinab Mansour from Cairo University also sent me helpful comments. I learned an enormous amount of mathematics from talking to and working with Richard Askey, to whom I am eternally grateful. I am also indebted to George Andrews for personally helping me on many occasions and for his work which inspired parts of my research and many parts of this book. The book by Gasper and Rahman (Gasper & Rahman, 1990) has been an inspiration for me over many years and I am happy to see the second edition now in print (Gasper & Rahman, 2004). It is the book I always carry with me when I travel, and I "never leave home without it." I learned a great deal of mathematics and picked up many ideas from collaboration with other mathematicians. In particular I thank my friends Christian Berg, Yang Chen, Ted Chihara, Jean Letessier, David Masson, Martin Muldoon, Jim Pitman, Mizan Rahman, Dennis Stanton, Galliano Valent, and Ruiming Zhang for the joy of having them share their knowledge with me and for the pleasure of working with them. P. G. (Tim) Rooney helped me early in my career, and was very generous with his time. Thanks, Tim, for all the scientific help and post-doctorate support.

This book was mostly written at the University of South Florida (USF). All the typesettng was done at USF, although during the last two years I was employed by the University of Central Florida. I thank Marcus McWaters, the chair of the Mathematics Department at USF, for his encouragement and continued support which enabled me to complete this book. It is my pleasure to acknowledge the enormous contribution of Denise L. Marks of the University of South Florida. She was always there when I needed help with this book or with any of my edited volumes. On many occasions, she stayed after office hours in order for me to meet deadlines. Working with Denise has always been a pleasure, and I will greatly miss her in my new job at the University of Central Florida.

In closing I thank the staff at Cambridge University Press, especially David Tranah, for their support and cooperation during the preparation of this volume and I look forward to working with them on future projects.

Orlando, FL
April 2005

Mourad E.H. Ismail