

Dimensional Analysis

Dimensional Analysis

Practical Guides in Chemical Engineering

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DEDICATION

To Sinta Worstell
Wife and best friend of 40 years

Introduction

1.1 PROCESS DEVELOPMENT

We generally classify chemical processes by their size: laboratory, pilot plant, or commercial. Process development entails moving technology from the laboratory to the commercial plant. Process development usually begins with an idea that is later successful in laboratory experiments. The proven idea then moves into a pilot plant and if successful there, progresses to a purpose-built commercial plant or into an existing commercial plant. Process support involves moving from an existing commercial plant into a laboratory or pilot plant to solve a problem. We may solve that problem by a small change to the process or with a major overhaul of the process, but, in either case, the solution is first tested in a laboratory or pilot plant, and then confirmed in a test run at the commercial plant.

Process development and process support involve identifying the variables pertinent to and controlling the chemical process, then designing experiments to establish the functional relationship between the variable of interest; i.e., the dependent variable—the variable that will make us money or is costing us money—and the independent variables of the process. These experimental programs must be doable and within the financial ability of the organization sponsoring them.

A good portion of such an experimental program involves collecting information that allows us to move the process from its current size to the next larger size. We call moving from one size to another “scaling.” Thus, process development involves upscaling and process support utilizes downscaling and upscaling. Upscaling involves starting with an idea, then proceeding from laboratory to commercial plant. Downscaling involves starting at a commercial plant and conducting laboratory experiments or operating a pilot plant to mimic the commercial plant problem, then upscaling the solution into the commercial plant.

At some point in your chemical engineering career, you will be asked either to develop a process or to provide technical support to an existing process. You are at that point in your career; otherwise, you would not be reading this book.

There are three methods for developing and upscaling a chemical process:

1. Build successively larger plants until you reach a defined commercial size.
2. Derive and solve the various conservation and transport equations describing your process, then use them to size the commercial process.
3. Establish the empirical relationships between the variables of your process, then use similarity to size the commercial process.

During the first 65–75 years of chemical engineering, we employed the first method when developing and upscaling a chemical process. There are several reasons why we used this method. First, during those years, most chemical processes were new and novel and chemical engineers possessed little information about them, particularly with regard to their safety. Therefore, chemical engineers incrementally increased, stepwise, the size of the process and established the operating conditions and monitored the interaction of the chemicals at each completed step. Second, chemical engineers used each successively larger processing unit as an analog computer, sampling the contents of each unit, then plotting the resulting data to obtain the solution to the differential equations describing the chemical process. Third, during those years, chemical and metallurgical engineers had to develop new materials of construction to meet the specifications of their new chemical processes, particularly with regard to high pressure, high temperature, and corrosion resistance. Fourth, physical property databases were rudimentary during those years. In many cases, intermediate-sized processing units; i.e., pilot plants, were built primarily to measure the physical properties of the chemical components comprising the new process. This process development method, while having advantages, is capital intensive, time consuming, and operationally expensive.

The second method arose with the advent of digital computing during the mid-1940s. Digital computing extends the hope that we can step directly from laboratory-sized equipment to commercial-sized equipment via calculation. The realization of this hope requires extensive, robust databases. Thus, the effort during the third and fourth quarters of the twentieth century to establish the physical properties of a wide variety of chemicals and to develop methods for estimating, via calculation, the physical properties of all chemicals. Also, during these

years, computing power doubled many times and software became evermore user-friendly. But, even with these advances, the conservation and transport equations for a given chemical process remain difficult to solve numerically. Much of this difficulty arises from the “stiff” differential equations describing the chemical process. Stiff means some of the differential equations have characteristic times much smaller than the other differential equations.¹ Numerical solutions are also difficult to obtain for catalyzed chemical processes. In a catalyzed chemical process, the catalyst is present at parts per million levels while the reactants are present at moles per liter levels. The same constraint occurs when impurities or by-products are included in a digital model. Again, impurities and by-products are present at parts per million levels while reactants and products are present at moles per liter levels. Such sets of differential equations demonstrate the same characteristics of stiff sets of differential equations; namely, numerical calculations will not “close,” will not approach a stable result. Today, it is possible to design portions of a chemical process directly from laboratory data, distillation showing the most success, but we are far from designing a complete commercial-sized chemical process via calculation.

Downsizing occurs when a commercial plant exists but a pilot plant for the chemical process does not exist. This situation occurs quite frequently for well-established commercial products, particularly commodity products. Unfortunately, processing issues still arise in such commercial plants, issues which require study in a nonexistent pilot plant. In such cases, you will be asked to design a pilot plant that mimics the commercial plant in order to develop solutions to commercial plant problems. Downsizing a chemical process sounds easy, until you try it. Traditionally, we have simply built a miniature “look-alike” plant when downsizing a commercial plant. This approach depends upon “luck” to reproduce the problem requiring solution. If we alter the controlling regime of the process due to downsizing, then we spend time, capital, and incur operating expense solving a problem related to the pilot plant rather than solving the commercial plant problem. This situation occurs many more times than we care to admit. To successfully downsize a chemical process, we must first identify the regime, either momentum transfer, energy transfer, or mass transfer, causing the problem in the commercial plant, then design the downsized unit to exactly mimic that regime. Otherwise, we expend considerable effort solving an unrelated problem. Due to misidentifying the controlling

regime, accurately downsizing a commercial chemical process is more difficult than upscaling it. Thus, downsizing is just as difficult as upscaling a chemical process, if not more so. Chemical engineers have applied digital computing to downsizing also; the success rate is about equivalent to that for upscaling a chemical process.

But, what can we do to successfully develop and support, and subsequently scale, a chemical process if its conservation laws cannot be solved analytically and if the differential equations comprising those laws are “stiff”; i.e., cannot be solved numerically? We could revert to our historic procedure for developing a chemical process; namely, build successively larger processing units until we reach the commercial-sized plant. Today’s global economy, however, puts severe constraints upon this particular method of process development and process support.

1.2 TWENTIETH-CENTURY REVOLUTIONS

Three twentieth-century revolutions provide the foundation of our global economy. The transportation revolution includes the invention of the box container, the development of large transport ships with evermore efficient power plants, and the development of highly efficient off-loading facilities. Shipping costs have plummeted during the last 30–40 years because of these technologies. Result: industrial product can be shipped globally at low cost, thereby increasing commercial competition.

The information revolution involves the advent of electronic communications and the Internet. Both have revolutionized commercial marketing, thereby providing every company access to every market ... globally. Thus, every company faces increased competition. With so much competition, the company introducing a process or product to the market makes the most money. In other words, the time from invention to commercialization must be short; otherwise, someone else will beat you to the marketplace and realize your potential profit. Time has become money.

Lastly: the financial revolution, which is based on electronic communications and the Internet. Today, financial markets never close ... they have become one that simply follows the sun as the earth as it rotates on its axis. Thus, there is severe competition for your investment dollars. This competition limits the capital available for investment in process development and process support; i.e., for building pilot plants.

In summary, low transportation costs means product will be manufactured in low-wage countries; thus, the drive to reduce operating costs at all commercial levels and process development expenditures. Competition for capital, in the form of return on investment, means future projects must have returns equal to or greater than those available in the financial market. Thus, the drive to control capital investment in pilot plants during process development. And, lastly, the advent of the Internet means the time from invention to commercialization; i.e., “cycle time,” must be reduced. Hence, the drive to minimize the time required to develop a process.

1.3 DIMENSIONAL ANALYSIS

So, how can we develop a chemical process in light of these constraints? Fortunately, there is a mathematical procedure available for upscaling and downscaling chemical processes that does not involve analytically or numerically solving the relevant conservation laws. That mathematical procedure is Dimensional Analysis.

When we upscale or downscale a chemical process, we use conservation laws that describe the changes imposed upon it. Such equations are written descriptively as

$$\text{accumulation} = \text{input} - \text{output} + \text{generation}$$

for a given chemical species. Each term in this descriptive equation contains physical quantities, such as constants, parameters, and variables. Constants fluctuate or vary the least, as their name implies. Constants do not change or vary within the space and during the time we use them. Such constants are inherent in the functioning of the universe. Some common constants are the speed of light in a vacuum, the gravitational constant, Boltzmann’s constant, and Planck’s constant. Defined constants, such as the permeability of vacuum, are exact. The majority of constants, however, are measured. The speed of light in vacuum and the gravitational constant are examples of measured constants.

Parameters are physical quantities that are unchanging in the context of a physical or chemical process. Parameters are not inherent to the functioning of our universe. They are inherent to the functioning of a given physical or chemical process. Thus, parameters change from one physical or chemical process to another.

The physical quantities that vary the most in a process are, as their name implies, variables. Variables come in two flavors: independent and dependent. Independent variables specify and define physical or chemical processes. Independent variables identify distinguishable requirements, those pertinent interactions between us and the physical or chemical process.² Thus, we can change independent variables directly through our interaction with the physical or chemical process. Dependent variables respond to our interaction with the process, to our changing an independent variable.

Dimensional Analysis provides relationships between the dependent variable and a minimum number of independent variables. As chemical engineers, we are introduced to Dimensional Analysis via Lord Rayleigh's Method of Indices and nondimensionalization of differential equations.

Dimensional homogeneity forms the foundation of Lord Rayleigh's Method of Indices. Dimensional homogeneity stipulates that the dimensions on either side of an equality sign must be the same.³ Lord Rayleigh's Method of Indices is best understood through example.⁴ Consider isothermal flow of a viscous Newtonian fluid: what variables influence it and how are they related? From centuries of study, we know that isothermal fluid flow in a pipe depends upon fluid velocity, pipe diameter, fluid density, fluid viscosity, fluid surface tension, if a free surface exists in the pipe during flow, and the force impinging the fluid.

Our first task when undertaking a Dimensional Analysis is to designate the fundamental dimensions we plan to use.⁵ For this problem, we will designate length (L), mass (M), and time (T) as our fundamental dimensions. Note that these fundamental dimensions are independent of each other; thus, they can be multiplied or divided by each other. In other words, they can yield "derived" dimensions, such as velocity (L/T) or force (ML/T²) and so on.

The above variables in terms of fundamental dimensions and derived dimensions are: velocity, v [L/T]; linear dimension, L [L]; force, F [ML/T²]; density, ρ [M/L³]; viscosity, μ [M/LT]; surface tension, σ [M/T²]; and, acceleration of gravity, g [L/T²]. The bracketed terms identify the fundamental dimensions describing each variable. For this example, we assume the mass in force and the mass in density, viscosity, and surface tension are measured in the same manner. In other

words, we will not encumber ourselves with a force mass (M_F) and a weight mass (M_W).⁶ Our long history of studying isothermal pipe flow indicates that the flow equation can be expressed as a function such as

$$f(v, L, F, \rho, \mu, \sigma, g) = 0$$

where $f()$ designates an unknown function. If we identify force F as the dependent variable, then the above function becomes

$$\kappa f(v, L, \rho, \mu, \sigma, g) = F$$

where κ is a constant to be determined experimentally.

From dimensional homogeneity, the above function can be expanded as a power series; thus

$$F = \kappa(v)^a(L)^b(\rho)^c(\mu)^d(\sigma)^e(g)^h$$

where a, b, c, \dots, h represent unknown powers or indices. In terms of fundamental and derived dimensions, the above power law becomes

$$[MLT^{-2}] = \kappa[LT^{-1}]^a[L]^b[ML^{-3}]^c[L^{-1}MT^{-1}]^d[M T^{-2}]^e[LT^{-2}]^h$$

Equating the powers or indices for each fundamental dimension yields

$$M:1 = c + d + e$$

$$L:1 = a + b - 3c - d + h$$

$$T: -2 = -a - d - 2e - 2g$$

Thus, we have six unknowns and three equations. Therefore, we can solve for three unknowns in terms of the remaining three unknowns. Solving for a, b , and c , we obtain

$$a = 2 - d - 2e - 2g$$

$$b = 2 - d - e + g$$

$$c = 1 - d - e$$

Substituting the above into the power law expression for F yields

$$F = \kappa(v)^{(2-d-2e-2g)}(L)^{(2-d-e+g)}(\rho)^{(1-d-e)}(\mu)^d(\sigma)^e(g)^h$$

Grouping similar powers or indices gives

$$F = \kappa(v^2 L^2 \rho) f \left\{ \left(\frac{vL\rho}{\mu} \right)^d, \left(\frac{v^2 L \rho}{\sigma} \right)^e, \left(\frac{v^2}{Lg} \right)^h \right\}$$

Rearranging the above equation yields

$$\frac{F}{v^2 L^2 \rho} = \kappa f \left\{ \left(\frac{vL\rho}{\mu} \right)^d, \left(\frac{v^2 L \rho}{\sigma} \right)^e, \left(\frac{v^2}{Lg} \right)^h \right\}$$

Note that each term in the above equation is dimensionless. The independent dimensionless terms are

$$\left(\frac{vL\rho}{\mu} \right)^d, \left(\frac{v^2 L \rho}{\sigma} \right)^e, \left(\frac{v^2}{Lg} \right)^h$$

We traditionally call such terms dimensionless parameters.⁷ The dependent dimensionless term is

$$\frac{F}{v^2 L^2 \rho}$$

We traditionally call such terms dimensionless coefficients.⁷

Note that the indices d , e , and h on the dimensionless parameters are undefined, as is the constant κ . We can assume values for d , e , and h so long as our assumptions ensure that the dimensionless parameters are independent of each other. In the past, engineers have assumed the dimensionless parameters are independent of each other if each parameter contains a unique variable. In the above equation, μ , σ , and g provide a unique variable in each dimensionless parameter to the right of the equality sign. Thus, these dimensionless parameters are independent of each other, which allows us to assume values for d , e , and h . Let us assume

$$d = e = h = 1$$

The function above then becomes

$$\frac{F}{v^2 L^2 \rho} = \kappa f \left\{ \left(\frac{vL\rho}{\mu} \right), \left(\frac{v^2 L \rho}{\sigma} \right), \left(\frac{v^2}{Lg} \right) \right\}$$

which represents the solution to our problem in functional notation. We must perform experiments to determine the functional relationship between the dimensionless coefficient and the dimensionless parameters.

If we know one variable is linearly dependent upon another variable, we need to perform only three experiments to determine the slope

and intercept of the function. In general, however, we do not know whether a given variable depends linearly on another variable or not. In such cases, we must perform at least five experiments to determine the curvature of the function or to determine whether the function is curved or oscillatory.⁸ Thus, for multivariable functions, the number of experiments to define the function grows rapidly because we must perform five experiments per independent variable while maintaining all other variables constant. Therefore, the total number of experiments $N_{\text{Expts}}^{\text{Total}}$ required to define a function is

$$N_{\text{Expts}}^{\text{Total}} = N_{\text{Expts}}^{M_{\text{IndepVar}}}$$

where N_{Expts} is the number of experiments to be performed per independent variable and M_{IndepVar} is the number of independent variables in the function.

For this example, the dimensionless function is

$$\frac{F}{v^2 L^2 \rho} = \kappa f \left\{ \left(\frac{v L \rho}{\mu} \right)^d, \left(\frac{v^2 L \rho}{\sigma} \right)^e, \left(\frac{v^2}{L g} \right)^h \right\}$$

Thus, we have one dimensionless coefficient and three dimensionless parameters. The total number of experiments required to define this function is

$$N_{\text{Expts}}^{\text{Total}} = N_{\text{Expts}}^{M_{\text{IndepVar}}} = 5^3 = 125$$

If we solve this example dimensionally, the solution function is

$$F = \kappa * f(v, L, \rho, \mu, \sigma, g)$$

and $N_{\text{Expts}}^{\text{Total}}$ is 5^6 or 15,625. By using Dimensional Analysis to determine the solution function for this example, we reduce the required number of experiments by a factor of 125, thereby greatly reducing the time and cost of defining the function.

The second Dimensional Analysis method we are taught utilizes our knowledge of the conservation laws; i.e., the differential equations, describing the physical or chemical process we are investigating. In most cases, we know these differential equations and we can inscribe them on paper. Unfortunately, we cannot solve them readily. If we could solve them, then we could readily upscale from laboratory-sized equipment to commercial-sized equipment via calculation. The fact

that we cannot solve such equations does not, however, mean we cannot use them. We can convert such dimensional differential equations into nondimensional differential equations and, by doing so, identify the terms that control the physical or chemical process under investigation.⁹

Let us consider, again, the isothermal flow of a viscous Newtonian fluid. For constant fluid density and fluid viscosity, the three-dimensional equation of change is

$$\rho \left(\frac{\partial u}{\partial t} \right) + \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g \cos \alpha_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

where ρ is fluid density (kg/m^3); μ is fluid viscosity (kg/m/s); t is time (s); u is fluid velocity in the x -direction (m/s); v is fluid velocity in the y -direction (m/s); w is fluid velocity in the z -direction (m/s); g is the acceleration of gravity (m/s^2); α_x is the angle at which the fluid flows in the x -direction; p is pressure ($\text{kg/s}^2 \text{ m}$); and, x , y , and z are directed lengths (m). We could have used pounds, feet, and seconds as our units since the magnitude of the measurement units; i.e., the system of units, does not impact the relationships derived via Dimensional Analysis.^{10,11}

If we write the above equation of change without operators and in one dimension, we have

$$\begin{array}{cccccc} \frac{\rho u}{t} + \frac{\rho u^2}{x} = \rho g - \frac{\Delta p}{x} + \frac{\mu u}{x^2} \\ 1 \quad \quad 2 \quad \quad 3 \quad \quad 4 \quad \quad 5 \end{array}$$

Each of the terms in the above equation represents a force. Term 1 is the force required to accelerate a unit volume of fluid during unsteady flow. Term 2 describes the change in momentum across a unit cross-sectional area perpendicular to the direction of flow. Term 3 represents the body forces experienced by a unit volume of fluid. Term 4 is the static pressure gradient within the unit fluid volume. And, lastly, Term 5 describes the reaction of the unit volume of fluid to applied shear force, which is viscous resistance. We can convert the above dimensional equation into a dimensionless equation by dividing each term by Term 5.¹¹ In other words, divide each force term by the viscous resistance or viscous force. Doing so yields

$$\left(\frac{x^2}{\mu u}\right)_1 \left(\frac{\rho u}{t}\right) + \left(\frac{x^2}{\mu u}\right)_2 \left(\frac{\rho u^2}{x}\right) = \left(\frac{x^2}{\mu u}\right)_3 (\rho g) - \left(\frac{x^2}{\mu u}\right)_4 \left(\frac{\Delta p}{x}\right) + 1$$

Multiplying Term 3 by one; i.e., by u/u , and multiplying Term 4 by one; i.e., $\rho u/\rho u$, then rearranging each term yields

$$\left(\frac{\rho x u}{\mu}\right)_1 \left(\frac{x}{u t}\right) + \left(\frac{\rho x u}{\mu}\right)_2 = \left(\frac{\rho x u}{\mu}\right)_3 \left(\frac{x g}{u^2}\right) - \left(\frac{\rho x u}{\mu}\right)_4 \left(\frac{\Delta p}{\rho u^2}\right) + 1$$

One parenthetical group is common to all the terms in the above equation; it is

$$\frac{\rho x u}{\mu}$$

The above dimensionless parameter is ubiquitous to the analysis of fluid flow. We call it the Reynolds number, in honor of Osborne Reynolds, who began a series of experiments into the nature of fluid flow in 1883. His first set of experiments identified two flow regimes: laminar flow and turbulent flow. Reynolds demonstrated in a second set of experiments conducted during the latter half of the 1880s that flow regime depends upon the above ratio. If $\rho x u/\mu < 2100$, then the flow regime is laminar; if $\rho x u/\mu > 4000$, then the flow regime is turbulent. If $2100 < \rho x u/\mu < 4000$, then the flow regime is unpredictable: it will either be laminar or turbulent, depending upon many variables beyond our control.¹² Note that the Reynolds number is the ratio of inertial forces to viscous forces.¹³

The second parenthetical group in Term 1, x/ut , is a modified Strouhal number.^{14,15} The Strouhal number describes oscillating flow mechanisms.

The second parenthetical group in Term 3 is the inverse of the Froude number. The Froude number is

$$\frac{u^2}{xg}$$

which is the ratio of inertial forces to gravitational forces.

The second parenthetical group in Term 4 is the “pressure coefficient” or Euler number. It is the ratio of pressure forces to inertial forces.¹⁶ All these ratios, except the Strouhal number, have a common feature: they are the ratio of a driving force to a resisting force.

Returning to our problem, the last equation above indicates that the solution is the function

$$f\left\{\left(\frac{\rho xu}{\mu}\right), \left(\frac{x}{ut}\right), \left(\frac{xg}{u^2}\right), \left(\frac{\Delta p}{\rho u^2}\right)\right\} = 0$$

Identifying $\frac{\Delta p}{\rho u^2}$ as the dimensionless coefficient allows us to write

$$\kappa g\left\{\left(\frac{\rho xu}{\mu}\right), \left(\frac{x}{ut}\right), \left(\frac{xg}{u^2}\right)\right\} = \frac{\Delta p}{\rho u^2}$$

as the solution to the problem. We must then perform experiments to determine the functional relationship between the dimensionless coefficient and the dimensionless parameters and to determine the value of κ .

Both these methods have problems. With regard to the Method of Indices, the number of unspecified indices grows as the number of constants, parameters, and variables describing the process increases. Performing the necessary algebra to identify the dimensionless parameters quickly becomes difficult and becomes a bookkeeping issue. But, more importantly, we must ensure, after identifying the dimensionless parameters, that they are linearly independent of each other.

With regard to the second method, we must have a complete set of differential equations and, again, we must determine that each of the resulting dimensionless parameters is linearly independent of all others.

What is not taught during these brief overviews of Dimensional Analysis is the concept of dimensions, systems of units and what units represent, and why the Method of Indices works. We must fully understand these points before we can recognize the power of Dimensional Analysis.

And lastly, what chemical engineers need is a procedure for doing Dimensional Analysis that greatly reduces the likelihood of making an algebraic error, that simplifies indices bookkeeping, and that guarantees the linear independence of the resulting dimensionless parameters. This book provides such a procedure.

1.4 SUMMARY

In this chapter, we discussed the historical procedures used by chemical engineers when developing a new process or modifying an existing process. As much as we desire to develop a process using the three conservation laws, we are unable to do so for a variety of reasons. However, we can use Dimensional Analysis and experimentation to define the functional relationships of the variables pertinent to a process. We briefly explained the two most common methods of Dimensional Analysis taught to chemical engineers and we discussed their shortcomings.

We also demonstrated in this chapter how Dimensional Analysis can reduce process development cycle time and costs. This feature of Dimensional Analysis is important in light of the three revolutions that occurred during the second half of the twentieth century. Those revolutions were in transportation, communication, and finance.

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CHAPTER 2

History of Dimensional Analysis

2.1 PRE JOSEPH FOURIER

The concept of dimension is as old as Greek mathematics, but the use of dimension as an analytical tool is relatively modern. Greek mathematics, i.e., geometry, is based on length and dimensionless angle. The Greeks did not consider the implications of dimension since all their mathematical manipulations involved only lengths and angles.

When our earliest ancestors learned to count is unknown, but it surely began shortly after they realized they had fingers and toes. With those fingers and toes, the rules of pure number manipulation came to light. Thus, by the time history began, our ancestors knew how to manage pure numbers; they knew the rules of arithmetic.

With the development of algebra, higher mathematics freed itself from geometry. In algebra, numbers can represent physical quantities, which have dimensions. Attaching dimensional information to numbers negates the rules of arithmetic, unless we observe certain restrictions.¹

The first to discuss the concept of dimension was Johannes de Mures (c.1290–c.1355), a French philosopher, astronomer, mathematician, and music theorist. He wrote about products and quotients possessing different dimensions. However, his work on products and quotients made no lasting impression on the development of science.

Descartes (1596–1650) may have been the first natural philosopher and mathematician to realize that derived dimensions exist, such as Force.² According to Descartes, “[t]he force to which I refer always has two dimensions, and it is not the force that resists (a weight) which has one dimension”.³ However, Descartes was not hindered by mathematical operations that produced dimensionally impossible results. For Descartes, dimensional correctness did not determine the correctness of a given result.

Sir Issac Newton (1642–1727) recognized the concept of derived dimensions: “I call any quantity a genitum which is not made by

addition or subtraction of divers parts, but is generated or produced in arithmetic by multiplication, division, or extraction of the root of any term whatsoever . . .”⁴ Gottfried Leibniz (1646–1716) also recognized the concept of derived dimensions, no doubt to Sir Issac’s chagrin: “. . . action . . . is as the product of the mass multiplied by space and velocity, or as the time multiplied by *vis viva*.”⁴

The eighteenth century witnessed great advances in analysis of physical phenomena; however, little thought was given to dimensions. Leonhard Euler (1707–1783) was the only natural philosopher and mathematician to make comment on dimensions during that momentous century. In fact, Euler demonstrated a preoccupation about the meaning of physical relationships. In 1736, Euler published *Mechanica* in which he showed that the dimension of n in the equation

$$A \times dv = np \times dx$$

depended on the dimensions of A (mass) and p (force). This observation by Euler indicates that he understood the need for unit homogeneity; that is, the units left of an equal sign must be the same as those units to the right of the same equal sign. Euler further discussed dimensions in his *Theoria motus corporum solidorum seu rigidorum* published in 1755. In this book, Euler devoted a chapter to questions of units and homogeneity. Unfortunately, his writings about dimension made little impression upon the community of mathematicians and natural philosophers of the time.

2.2 POST JOSEPH FOURIER

Little, if any, discussion of dimensions occurred after Euler’s *Theoria* until 1822 when Joseph Fourier published the third edition of his *Analytical Theory of Heat*. Fourier makes no mention of dimension in either the first edition of his book, published in 1807, or the second edition, published in 1811. However, in the 1822 edition, Fourier specifically states that any system of units can be used to study a physical process, so long as the chosen system of units is consistent. He also states that mathematical equations used to describe physical processes must demonstrate homogeneity: the units on either side of an equal sign must be the same. Fourier used the concept of homogeneity to check his mathematical manipulations. He clearly states in the 1822 edition that a natural philosopher should use unit homogeneity as a