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Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces

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Preface

The aim of this book is to give a self-contained introduction to the theory of infinite-dimensional systems theory and its applications to port-Hamiltonian systems.

The field of infinite-dimensional systems theory has become a well-established field within mathematics and systems theory. There are basically two approaches to infinite-dimensional linear systems theory: an abstract functional analytical approach and a PDE approach. There are excellent books dealing with infinite-dimensional linear systems theory, such as (in alphabetical order) Bensoussan, Da Prato, Delfour and Mitter [6], Curtain and Pritchard [9], Curtain and Zwart [10], Fattorini [17], Luo, Guo and Morgul [40], Lasiecka and Triggiani [34, 35], Lions [37], Lions and Magenes [38], Staffans [51], and Tucsnak and Weiss [54].

Many physical systems can be formulated using a Hamiltonian framework. This class contains ordinary as well as partial differential equations. Each system in this class has a Hamiltonian, generally given by the energy function. In the study of Hamiltonian systems it is usually assumed that the system does not interact with its environment. However, for the purpose of control and for the interconnection of two or more Hamiltonian systems it is essential to take this interaction with the environment into account. This led to the class of port-Hamiltonian systems, see [56, 57]. The Hamiltonian/energy has been used to control a port-Hamiltonian system, see e.g. [4, 7, 21, 43]. For port-Hamiltonian systems described by ordinary differential equations this approach is very successful, see the references mentioned above. Port-Hamiltonian systems described by partial differential equations is a subject of current research, see e.g. [14, 28, 33, 41].

In this book, we combine the abstract functional analytical approach with the more physical approach based on Hamiltonians. For a class of linear infinite-dimensional port-Hamiltonian systems we derive easily verifiable conditions for well-posedness and stability.

The material of this book has been developed over a series of years. Javier Villegas [58] studied in his PhD-thesis a port-Hamiltonian approach to distributed parameter systems. We are grateful to Javier Villegas that we could include his results into the book. The first setup of the book was written for a graduate course on control of distributed parameter systems for the Dutch Institute of Systems and Control (DISC) in the spring of 2009 which was attended by 25 PhD students. This material was adapted for the CIMPA-UNESCO-Marrakech School on Control and Analysis for PDE in May 2009. In 2010-2011 we were the virtual lecturers of the 14th Internet Seminar on Infinite-dimensional Linear Systems Theory. More than 300 participants attended this virtual course and a wikipage was used to discuss the material and to post typos and comments. For this course we decided to add extra chapters on finite-dimensional systems theory, and to make the material in the later chapters more accessible.

We are indebted to the help from many colleagues and friends. We are grateful to the participants of the DISC-course, the CIMPA-UNESCO-Marrakesch School and the 14th Internet Seminar for their useful comments and questions. Large parts of the manuscript have been read by our colleagues Mikael Kurula (Twente) and Christian Wyss (Wuppertal), who made many useful comments for improvements.

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Birgit Jacob and Hans Zwart,
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Wuppertal and Twente

Contents

List of Figures

Chapter 1

Introduction

In this chapter we provide an introduction to the field of mathematical systems theory. Besides examples we discuss the notion of feedback and we answer the question why feedback is useful. However, before we start with the examples we discuss the following picture, which can be seen as the essence of systems theory. In

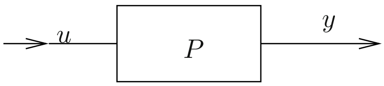

Figure 1.1: A system

systems theory, we consider models which are in contact with their environment. In the above picture, P denotes the to-be-studied-model which interacts with its environment via u and y. u and y are time signals, i.e., functions of the time t. The function u denotes the signal which influences P and y is the signal which we observe from P. u is called the *input* or *control* and y is called the *output*. The character P is chosen, since it is short for plant, think of chemical or power plant. To obtain a better understanding for the general setting as depicted in Figure 1.1, we discuss several examples in which we indicate the input and output.

Regarding these examples, we should mention that we use different notations for derivatives. In the best tradition of mathematics, we use $\dot{f}$, $\frac{df}{dt}$, and $f^{(1)}$ to denote the first derivative of the function f. Similarly for higher derivatives.

1.1 Examples

Example 1.1.1. Newton's second law states that the force applied to a body produces a proportional acceleration; that is

$$F(t) = m\ddot{q}(t). \tag{1.1}$$

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Here $q(t)$ denotes the position at time t of the particle with mass m, and $F(t)$ is the force applied to it. Regarding the external force $F(t)$ as our input $u(t)$ and choosing the position $q(t)$ as our output $y(t)$, we obtain the differential equation

$$\ddot{y}(t) = \frac{1}{m} u(t), \qquad t \geq 0. \tag{1.2}$$

Thus the differential equation describes the behaviour "inside the box P", see Figure 1.1. Further, we can influence the system via the external force u and we observe the position y. In this simple example we clearly see that u is not the only quantity that determines y. The output also depends on the initial position $q(0)$ and the initial velocity $\dot{q}(0)$. They are normally not at our disposal to choose freely, and so they are also "inside the box".

Example 1.1.2. Consider the electrical network given by Figure 1.2. Here V denotes the voltage source, L_1, L_2 denote the inductance of the inductors, and C denotes the capacitance of the capacitor.

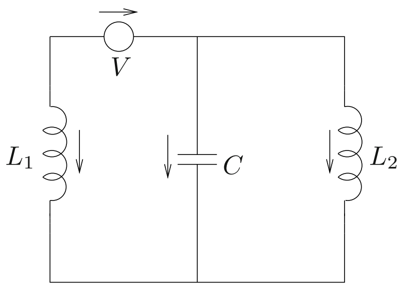

Figure 1.2: Electrical network

For the components in an electrical circuit, the following basic laws hold: the current I_L and voltage V_L across an inductor with inductance L is related via

$$V_L(t) = L \frac{dI_L}{dt}(t), \tag{1.3}$$

whereas the current I_C and voltage V_C of the capacitor with capacitance C are related via

$$I_C(t) = C \frac{dV_C}{dt}(t). \tag{1.4}$$

The conservation of charge and energy in electrical circuits are described by Kirchhoff's circuit laws. Kirchhoff's first law states that at any node in an electrical circuit, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Moreover, Kirchhoff's second law says that the directed sum of the electrical potential differences around any closed circuit must be zero.

Applying these laws to our example, we obtain the following differential equations:

$$L_1 \frac{dI_{L_1}}{dt}(t) = V_{L_1}(t) = V_C(t) + V(t), \tag{1.5}$$

$$L_2 \frac{dI_{L_2}}{dt}(t) = V_{L_2}(t) = V_C(t), \qquad \text{and} \tag{1.6}$$

$$C \frac{dV_C}{dt}(t) = I_C(t) = -I_{L_1}(t) - I_{L_2}(t). \tag{1.7}$$

We assume that we can only measure the current I_{L_1}, i.e. we define

$$y(t) = I_{L_1}(t).$$

Using (1.5), we obtain $L_1 y^{(1)}(t) = V_C(t) + V(t)$. Further, (1.7) then implies

$$L_1 C y^{(2)}(t) = -y(t) - I_{L_2}(t) + CV^{(1)}(t). \tag{1.8}$$

Differentiating this equation once more and using (1.6), we find

$$\begin{aligned} L_1 C y^{(3)}(t) &= -y^{(1)}(t) - \frac{dI_{L_2}}{dt}(t) + CV^{(2)}(t) = -y^{(1)}(t) - \frac{1}{L_2} V_C(t) + CV^{(2)}(t) \\ &= -y^{(1)}(t) - \frac{1}{L_2}\Big(L_1 y^{(1)}(t) - V(t)\Big) + CV^{(2)}(t), \end{aligned} \tag{1.9}$$

where we have used (1.5) as well. We regard the voltage supplied by the voltage source as the input u. Thus we obtain the following ordinary differential equation describing our system:

$$L_1 C y^{(3)}(t) + \left(1 + \frac{L_1}{L_2}\right) y^{(1)}(t) = \frac{1}{L_2} u(t) + C u^{(2)}(t). \tag{1.10}$$

Example 1.1.3. Suppose we have a mass m which can move along a line, as depicted in Figure 1.3. The mass is connected to a spring with spring constant k, which in turn is connected to a wall. Furthermore, the mass is connected to a damper whose (friction) force is proportional to the velocity of the mass by the constant r. The third force which is working on the mass is given by the external force $F(t)$.

Let $q(t)$ be the distance of the mass to the equilibrium point. Then by Newton's law we have that

$$m\ddot{q}(t) = \text{total sum of the forces.}$$

As the force of the spring equals $kq(t)$ and the force by the damper equals $r\dot{q}(t)$, we find

$$m\ddot{q}(t) + r\dot{q}(t) + kq(t) = F(t). \tag{1.11}$$

We regard the external force $F(t)$ as our input u and we choose the position as our output y. This choice leads to the differential equation

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t). \tag{1.12}$$

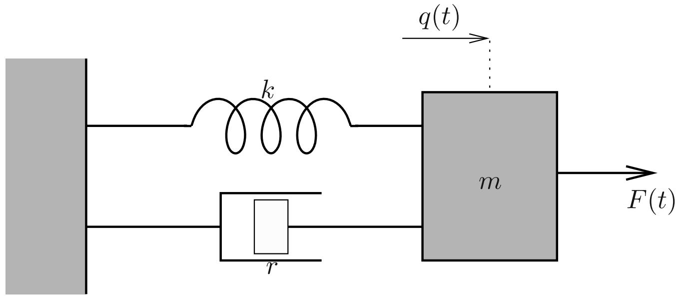

Figure 1.3: Mass-spring-system

Until now we have only seen examples which can be modelled by ordinary differential equations. The following examples are modelled via a partial differential equation.

Example 1.1.4. We consider the *vibrating string* as depicted in Figure 1.4. The

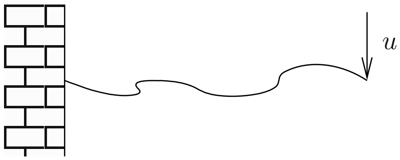

Figure 1.4: The vibrating string

string is fixed at the left-hand side and may move freely at the right-hand side. We allow that a force u may be applied at that side. The model of the (undamped) vibrating string is given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right) \tag{1.13}$$

where $\zeta \in [a, b]$ is the spatial variable, $w(\zeta, t)$ is the vertical displacement of the string at position ζ and time t, T is the Young's modulus of the string, and ρ is the mass density, which may vary along the string. This model is a simplified version of other systems where vibrations occur, as in the case of large structures, and it is also used in acoustics. The partial differential equation (1.13) is also known as the *wave equation*.

If the mass density and the Young's modulus are constant, then we get the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \tag{1.14}$$

where $c^2 = T/\rho$. This is the most familiar form of the wave equation.

In contrast to ordinary differential equations, we need boundary conditions for our partial differential equation (1.13) or (1.14). At the left-hand side, we put the position to zero, i.e.,

$$w(a,t) = 0 \tag{1.15}$$

and at the right-hand side, we have the balance of the forces, which gives

$$T(b)\frac{\partial w}{\partial \zeta}(b,t) = u(t). \tag{1.16}$$

There are different options for the output. One option is to measure the velocity at the right-hand side, i.e.,

$$y(t) = \frac{\partial w}{\partial t}(b,t). \tag{1.17}$$

Another option could be to measure the velocity at a point between a and b, or to measure the position of the wave, i.e., $y(t) = w(\cdot,t)$. Hence at every time instant, y is a function of the spatial coordinate.

We end this section with another well-known partial differential equation.

Example 1.1.5. The model of *heat conduction* consists of only one conservation law, that is, the *conservation of energy*. It is given as

$$\frac{\partial e}{\partial t} = -\frac{\partial}{\partial \zeta} J_Q, \tag{1.18}$$

where $e(\zeta,t)$ is the energy density and $J_Q(\zeta,t)$ is the heat flux. This conservation law is completed by two closure equations. The first one expresses the calorimetric properties of the material:

$$\frac{\partial e}{\partial T} = c_V(T), \tag{1.19}$$

where $T(\zeta,t)$ is the temperature distribution and c_V is the heat capacity. The second closure equation defines the heat conduction property of the material (Fourier's conduction law):

$$J_Q = -\lambda(T,\zeta)\frac{\partial T}{\partial \zeta}, \tag{1.20}$$

where $\lambda(T,\zeta)$ denotes the heat conduction coefficient. Assuming that the variations of the temperature are not too large, we may assume that the heat capacity and the heat conduction coefficient are independent of the temperature. Thus we obtain the partial differential equation

$$\frac{\partial T}{\partial t}(\zeta,t) = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T}{\partial \zeta}(\zeta,t)\right), \qquad \zeta \in (a,b),\ t \geq 0. \tag{1.21}$$

As for the vibrating string, the constant coefficient case is better known. This is

$$\frac{\partial T}{\partial t}(\zeta,t) = \alpha\frac{\partial^2 T}{\partial \zeta^2}(\zeta,t), \qquad \zeta \in (a,b),\ t \geq 0 \tag{1.22}$$

with $\alpha = \lambda/c_V$.

Again, we need boundary conditions for the partial differential equations (1.21) and (1.22). If the heat conduction takes places in a perfectly insulated surrounding, then no heat can flow in or out of the system, and we have as boundary conditions

$$\lambda(a)\frac{\partial T}{\partial \zeta}(a,t) = 0, \text{ and } \lambda(b)\frac{\partial T}{\partial \zeta}(b,t) = 0. \tag{1.23}$$

It can also be that the temperature at the boundary is prescribed. For instance, if the ends are lying in a bath with melting ice, then we obtain the boundary conditions

$$T(a,t) = 0, \text{ and } T(b,t) = 0. \tag{1.24}$$

As measurement we can take the temperature at a point $y(t) = T(\zeta_0,t), \zeta_0 \in (a,b)$. Another choice could be the average temperature in an interval (ζ_0,ζ_1). In the latter case we find

$$y(t) = \frac{1}{\zeta_1 - \zeta_0}\int_{\zeta_0}^{\zeta_1} T(\zeta,t)\,d\zeta.$$

As input we could control the temperature at one end of the spatial interval, e.g. $T(b,t) = u(t)$, or we could heat it in the interval (ζ_0,ζ_1). The latter choice leads to the partial differential equation

$$\frac{\partial T}{\partial t}(\zeta,t) = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T}{\partial \zeta}(\zeta,t)\right) + u(\zeta,t),$$

where we define $u(\zeta,t) = 0$ for $\zeta \notin (\zeta_0,\zeta_1)$.

1.2 How to control a system?

In the previous section we have seen that there are many models in which we can distinguish an input and an output. Via the input we have the possibility to influence the system. In particular, we aim to choose the input u such that y or all variables in the box behave as we desire. Note that the phrase "to behave as we desire" means that we have to make choices. These choices will be based on the type of plant we are dealing with. If P represents a passenger airplane, and y the height, we do not want y to go from 10 kilometers to ground level in one second. However, we would like that this happens in half an hour. On the other hand, if the plant represents a stepper, then very fast action is essential. A stepper is a device used in the manufacture of integrated circuits (ICs); it got its name from the fact that it moves or "steps" the silicon wafer from one shot location to another. This has to be done very quickly, and with a precision on nano-meters.

So generally, the control task is to find an input u such that y has a desired behaviour and we shall rarely do it by explicitly calculating the function u. More often we design u on the basis of y. Hence instead of open loop systems described by Figure 1.1, here given once more as Figure 1.5, we work with closed loop systems given by Figure 1.6. In order to "read" the latter picture, it is sufficient to know

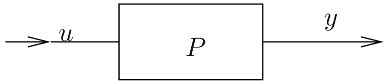

Figure 1.5: Our system

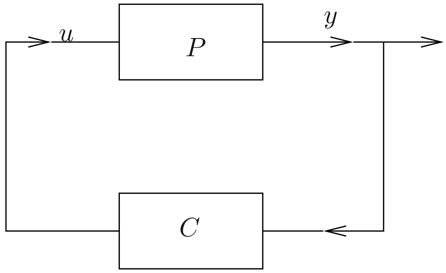

Figure 1.6: Feedback system

some simple rules. As before, by a rectangular block we denote a model relating the incoming signal to the outgoing signal. This could for instance be an ordinary or partial differential equation. Mathematically speaking one may see P and C as operators mapping the signal u to the signal y and vice versa. At a node we assume that the incoming signals are the same as the outgoing signals. The arrows indicate the directions of the signals.

As before we denote by P the system that we desire to control, and by C we denote our (designed) controller.

In this section we show that closed loop systems have in general better properties than open loop systems. We explain the advantages by means of an example. We consider the simple control problem of steering the position of the mass m to zero, see Example 1.1.1. Hence, the control problem is to design the force F such that, for every initial position and every initial velocity of the mass, the position of the mass is going to zero for time going to infinity. To simplify the problem even more we assume that the mass m equals 1, and we assume that we measure the velocity and the position, i.e.,

$$\ddot{q}(t) = u(t) \quad \text{and} \quad y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \tag{1.25}$$

In Exercise 1.2 the case $y(t) = q(t)$ is discussed.

In order to design the controller C for the plant described by (1.25) we proceed as follows. First we have to specify what we mean by "to behave as we desire", that is, we have to decide how and how fast the position should go to zero. It is quite common to choose an "ideal" differential equation such that the solutions have the desired behaviour. We consider the solutions of the differential equation

$$f^{(2)}(t) + 2f^{(1)}(t) + f(t) = 0. \tag{1.26}$$

The general solution of (1.26) is given by $f(t) = \alpha e^{-t} + \beta t e^{-t}$. This is to our satisfaction, and so we try to design a control u such that the position q is equal to a solution of the equation (1.26). If we choose

$$u(t) = \begin{bmatrix} -1 & -2 \end{bmatrix} y(t) = -2q^{(1)}(t) - q(t), \tag{1.27}$$

then the position of the mass indeed behaves in the same way as the solutions of (1.26). Note that for the design of this feedback law no knowledge of the initial position nor the initial velocity is needed. If we want to obtain the same behavior of the solution by an open loop control, i.e., if we want to construct the input as an explicit function of time, we need to know the initial position and the initial velocity. Suppose they are given as $q(0) = -2$ and $q^{(1)}(0) = 5$, respectively. Calculating the position as the solution of

$$q^{(2)}(t) + 2q^{(1)}(t) + q(t) = 0, \quad t \geq 0, \qquad q(0) = -2, \quad q^{(1)}(0) = 5,$$

we obtain

$$q(t) = -2e^{-t} + 3te^{-t}. \tag{1.28}$$

Thus by (1.27) we find

$$u(t) = -8e^{-t} + 3te^{-t}. \tag{1.29}$$

Applying this input to the system

$$q^{(2)}(t) = u(t) \tag{1.30}$$

would give the same behavior as applying (1.27).

We simulate the open loop system, i.e., (1.30) with $u(t)$ given by (1.29) and the closed loop system, i.e, (1.30) with $u(t)$ given by (1.27). The result is shown in Figure 1.7. We obtain that the simulation of the open loop system is worse than the one of the closed loop system. This could be blamed on a bad numerical solver, but even mathematically, we can show that the closed loop system behaves in a superior manner to the open loop system. We have assumed that we know the initial data exactly, but this will never be the case. So suppose that we have (small) errors in both initial conditions, but we are unaware of the precise error. Thus we apply the input (1.29) to the system

$$q^{(2)}(t) = u(t), \quad t \geq 0, \qquad q(0) = q_0, \quad q^{(1)}(0) = q_1,$$