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With 13 Figures

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It is a pleasure to dedicate this book to
Professor *I. N. Sneddon*, O.B.E., F.R.S.
It would not have come into existence had it
not been for his encouragement and advice
on general content.

Preface

The classical theories of Linear Elasticity and Newtonian Fluids, though triumphantly elegant as mathematical structures, do not adequately describe the deformation and flow of most real materials. Attempts to characterize the behaviour of real materials under the action of external forces gave rise to the science of Rheology. Early rheological studies isolated the phenomena now labelled as viscoelastic. Weber (1835, 1841), researching the behaviour of silk threats under load, noted an instantaneous extension, followed by a further extension over a long period of time. On removal of the load, the original length was eventually recovered. He also deduced that the phenomena of stress relaxation and damping of vibrations should occur. Later investigators showed that similar effects may be observed in other materials. The German school referred to these as "Elastische Nachwirkung" or "the elastic aftereffect" while the British school, including Lord Kelvin, spoke of the "viscosity of solids". The universal adoption of the term "Viscoelasticity", intended to convey behaviour combining properties both of a viscous liquid and an elastic solid, is of recent origin, not being used for example by Love (1934), though Alfrey (1948) uses it in the context of polymers.

The earliest attempts at mathematically modelling viscoelastic behaviour were those of Maxwell (1867) (actually in the context of his work on gases; he used this model for calculating the viscosity of a gas) and Meyer (1874). The model proposed by Meyer is generally associated with the names of Kelvin and Voigt who however made their contributions much later. These differential constitutive relations are discussed in section 1.6. Boltzmann (1874) proposed the general hereditary integral form of the constitutive relations which is the basis of most theoretical work on viscoelastic materials of the past three decades, though, historically, it was slow in gaining acceptance against the more specialized and cumbersome differential constitutive relations based on mechanical models, which essentially generalize the work of Maxwell and Meyer. Boltzmann gives the form applicable to isotropic bodies while Volterra (1909) gives the general anisotropic form. Volterra's theory of functionals (1959) is also at the basis of the modern formulation of Viscoelasticity in the general non-linear case. Further discussion of the early history of the subject may be found in the interesting article by Markovitz (1977), and in the book by Love (1934). Until the nineteen fifties, development of the subject was slow. The emergence into common use of a large variety of polymeric materials in the post-war years focussed increasing attention on the topic. References which focus on polymeric materials and their behaviour include Eirich (1956), Staverman and Schwarzl (1956), Ferry (1970) and (more recently) Doi and Edwards (1986). The application of the theory to metals has been surveyed by Zener (1948), while Arutyunyan (1952) and Bazant (1975) survey literature that treats concrete as an aging viscoelastic material.

Viscoelastic boundary value problems have been actively researched now for more than thirty years. During the nineteen fifties, attention centred mainly on the Classical Correspondence Principle and problems to which it was applicable (e.g. see Hunter (1960) and Lee (1960)). Extensions of this principle were discovered during the sixties. Also, several problems not covered by the classical form of the principle received attention. These were of two kinds, extending crack problems and contact problems where the load on the indentor is varying or the indentor is moving across the surface. These problems were of interest in the context of polymer fracture, rebound testing of polymers and the phenomenon of hysteretic friction, respectively. Methods were also developed to handle problems involving thermoviscoelastic behaviour where the dependence on temperatue is non-linear. All of this work presupposed that inertial effects could be neglected. Very little work on inertial boundary value problems was published up to the end of the sixties.

Comprehensive surveys of the application of viscoelastic stress analysis to design were prepared by Rogers (1965) and Lee (1966).

During the seventies, the work on non-inertial problems was consolidated. The main purpose of the present volume is to present a coherent, unified development of this topic, in particular of those problem classes which are not covered by the Classical Correspondence Principle. There has also been some progress on inertial problems. Typically however, to make progress on such problems it is necessary either to confine one's attention to the most idealized configurations or to introduce some approximation. Also, the mathematical techniques used have been generally rather sophisticated. We briefly discuss this work in the last chapter, and derive certain results by comparatively elementary methods.

The theory is developed without any serious attempt at mathematical rigour. However, we also avoid the use of merely heuristic arguments which are particularly common in the literature on fracture. The orientation of the book is applied mathematical, though with the ultimate aim of extracting physically interesting results. Certain required techniques and results, notably the statement and solution of the Hilbert problem and the use of Hilbert transforms, are discussed in several mathematical appendices. Short tables of integrals and other relations are included.

In chapter 1, the properties of the viscoelastic functions are explored in some detail. Also the boundary value problems of interest are stated. In chapter 2, the Classical Correspondence Principle and its generalizations are discussed. Then, general techniques, based on these, are developed for solving non-inertial isothermal problems. A method for handling non-isothermal problems is also discussed and in chapter 6 an illustrative example of its application is given. Chapter 3 and 4 are devoted to plane isothermal contact and crack problems, respectively. They utilize the general techniques of chapter 2. The viscoelastic Hertz problem and its application to impact problems are discussed in chapter 5. Finally in chapter 7, inertial problems are considered.

Exercises are scattered throughout the text, one of their main purposes being to allow the statement, without detailed derivation, of fairly standard or straightforward results. The equations occurring in these problems are numbered separately from the ordinary equations. They are distinguished by the letter "p" occurring after the number.

One of the authors (JMG) wishes to acknowledge gratefully two most pleasant periods spent at Simon Fraser University for the academic year 1983/84 and during the Spring of 1986. It was during these periods that most of his contribution to this work was made. He would also like to acknowledge the encouragement of P. O'Keefe, Head of Roads Division, J. Sheedy, Head of Road Construction Section and A. J. Curran, Head of Road Safety Section of the National Institute for Physical Planning and Construction Research over the years, and for their appreciation of the importance in some contexts, of a fundamental approach to applied research problems. The other author (GACG) acknowledges generous use of the facilities of the Dublin Institute for Advanced Studies particularly during his stay there during the academic year 1986/87.

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The Authors

Contents

1. Fundamental Relationships

This chapter deals with fundamental definitions, constitutive equations of a viscoelastic medium subject to infinitesimal strain, and the nature and properties of the associated viscoelastic functions. General dynamical equations are written down. Also, the boundary value problems that will be discussed in later chapters are stated in general terms. Familiar concepts from the Theory of Linear Elasticity are introduced in a summary manner. For a fuller discussion of these, we refer to standard references (Love (1934), Sokolnikoff (1956), Green and Zerna (1968), Gurtin (1972)). Coleman and Noll (1961) have shown that the theory described here may be considered to be a limit, for infinitesimal deformations, of the general (non-linear) theory of materials with memory.

1.1 Stress and Strain

The mechanical system under consideration is a continuum, subject to boundary forces which are in general time-dependent and also body forces, though the latter are usually neglected. These external forces generate stresses throughout the medium. Consider a typical point $\boldsymbol{r}$ within the medium, where $\boldsymbol{r}$ denotes the position vector with Cartesian co-ordinates (x, y, z) or (x_1, x_2, x_3)[1]. The stresses acting in the vicinity of $\boldsymbol{r}$ at time t are completely described by the stress tensor $\sigma(r, t)$ with Cartesian components $\sigma_{ij}(r, t)$, $i, j = 1, 2, 3$. This quantity is related to more fundamental concepts by the following observations. Lets ds denote any small surface in the medium or on its boundary. Let $\boldsymbol{n}$ be the unit normal on one side of it. Then the matter on this side of ds exerts a force per unit area, or stress, on the matter on the other side, given by the stress vector[2]

$$s_i = \sigma_{ij} n_j \ . \tag{1.1.1}$$

In particular, this formula relates stress tensor components on the boundary of the medium to the applied external stresses, and for this reason, will be important later. It follows from (1.1.1) that the force exerted by the medium or by the boundary stresses, on any volume V, is given by the surface integral

[1] Throughout the book, the subscripts 1, 2, 3 will denote x, y, z Cartesian components, respectively, of vectors and tensors.

[2] It will be assumed that the summation convention is in force here and in later sections, except where stated otherwise.

$$F_i = \int_B ds\, \sigma_{ij} n_j \ , \tag{1.1.2}$$

where B is the boundary of V, and the vector components n_j, $j = 1, 2, 3$ represent the outward normal at a given point on B.

The fact that the outward normal is used is a matter of convention. It implies for example that tensile stresses are positive and compressive stresses negative. Also, normal surface pressures into the medium along a co-ordinate axis generate a negative diagonal component of the stress tensor corresponding to that coordinate.

Problem 1.1.1: Show that if a body occupying the half-space $y > 0$ is subject to a surface shear stress along the positive x-axis, then this stress is equal to $-\sigma_{xy}$ on the surface.

The assumption that each small volume is in equilibrium, so that the resultant moment due to body and surface forces must vanish, gives, by means of a standard argument, that the stress tensor is symmetric. Viscoelastic materials with couple stresses, and therefore a non-symmetric stress tensor have been considered by Misicu (1963, 1964) and Eringen (1967). Also the requirement that the resultant of body and surface forces must vanish gives that the equations of motion take the form

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i \ , \tag{1.1.3}$$

where the $u_i(\boldsymbol{r}, t)$ are the components of the displacement vector $\boldsymbol{u}(\boldsymbol{r}, t)$ giving the displacement of the point $\boldsymbol{r}$ from its equilibrium position, and ϱ is the density of the medium. The vector components b_i in (1.1.3) are the contributions of body forces such as gravity. The space derivatives of displacements will be assumed to be sufficiently small that products of these quantities can be neglected compared to the quantities themselves. This restriction allows us to construct a linear theory. The state of deformation of the medium is then characterized by the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{r}, t)$ whose components are

$$\varepsilon_{ij}(\boldsymbol{r}, t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ , \qquad i, j = 1, 2, 3 \ . \tag{1.1.4}$$

On physical grounds, we expect a close relationship to exist between the components of stress and strain.

Note that the density ϱ in (1.1.3) will depend upon the state of deformation of the material. However, this correction may be ignored in the linear theory, provided that the acceleration term multiplying ϱ is assumed to be of the same order as the space derivatives of the displacements.

It must be remarked that the linear assumption may be especially restrictive for certain viscoelastic media, namely those which have low modulus. Many polymeric materials are in this category. For such materials, large deformations may occur even at relatively low stresses.

The traces of the tensors $\boldsymbol{\varepsilon}, \boldsymbol{\sigma}$ have special significance, since they are scalar quantities, and thus independent of the coordinate system. It is easy to demonstrate that

$$\mathrm{Tr}\{\boldsymbol{\varepsilon}\} = \frac{\partial u_i}{\partial x_i} = \varepsilon \tag{1.1.5}$$

is the volume strain or the change in volume per unit volume at a given point. The quantity

$$p = \tfrac{1}{3}\mathrm{Tr}\{\boldsymbol{\sigma}\} = \tfrac{1}{3}\sigma \tag{1.1.6}$$

is sometimes referred to as the hydrostatic stress, while

$$\boldsymbol{s} = \boldsymbol{\sigma} - p\boldsymbol{I} \ , \tag{1.1.7}$$

where $\boldsymbol{I}$ is the unit tensor, is the deviator stress tensor. This decomposition into hydrostatic and deviator stresses corresponds to the separation of volume and shear stresses.

We observe that strain is dimensionless and that stress has dimensions of force per unit area.

It is of interest to consider the rate of work done by the external and body forces on the medium. Let us denote this quantity by $\dot{E}$. We have

$$\dot{E} = \int_V b_i \dot{u}_i dv + \int_B \sigma_{ij} n_j \dot{u}_i ds \ , \tag{1.1.8}$$

where V is the total volume of the medium and B is its surface. The quantity dv is a volume element. Applying the Divergence Theorem to the second term and carrying out the differentiation gives that

$$\int_B \sigma_{ij} n_j \dot{u}_i ds = \int_V dv(\sigma_{ij,j}\dot{u}_i + \sigma_{ij}\dot{u}_{i,j}) \ , \tag{1.1.9}$$

where the subscript preceded by a comma indicates differentiation with respect to that space variable. It is easy to show that

$$\sigma_{ij}\dot{u}_{i,j} = \sigma_{ij}\dot{\varepsilon}_{ij} \ . \tag{1.1.10}$$

Using (1.1.3), we finally obtain

$$\dot{E} = \int_V (\varrho\,\dot{u}_i\ddot{u}_i + \sigma_{ij}\dot{\varepsilon}_{ij})dv \ . \tag{1.1.11}$$

We infer that the rate of increase of mechanical energy per unit volume at a point $\boldsymbol{r}$ at time t is given by

$$e = \frac{d}{dt}\left(\frac{1}{2}\varrho\,\dot{u}_i\dot{u}_i\right) + \sigma_{ij}\dot{\varepsilon}_{ij} = \dot{k} + \sigma_{ij}\dot{\varepsilon}_{ij} \ , \tag{1.1.12}$$

where $\dot{k}$ is the rate of change of kinetic energy per unit volume, with time.

We will show later that, in a viscoelastic medium, some of this energy supplied by external forces is stored and some is dissipated.

In order to give physical content to the theory, it is necessary to postulate constitutive relationships between the stress and strain tensors, so that complete knowledge of one determines the other. In this manner, we introduce into the theory the physical characteristics of the material under consideration.

1.2 One-dimensional Linear Viscoelasticity

We consider one-dimensional constitutive equations in some detail before moving on to the general case. In the context of boundary value problems, the one-dimensional case is of limited interest. However, if provides a simple framework in which to discuss a number of points that are of general importance. Also, of course, from the viewpoint of experimental material characterization, one-dimensional configurations are of great interest.

A particular one-dimensional problem is one in which every component of the stress and strain tensors is zero except for instance the xy components $\sigma_{12}, \varepsilon_{12}$. This would occur if a stress σ_{12} were imposed on a medium in which displacement takes place only in the y direction, depending on x alone – perhaps due to constraints. Following this model, we take it that the stress and strain depend only on x. Subscripts will be dropped.

The treatment of the material in this section is similar in many respects to that given by F. Williams (1975)

1.2.1 Linear Hereditary Constitutive Laws

The primary requirement which will be imposed here on the constitutive equation is that it be linear. This implies that the stress, for example, must be given as a linear functional of the strain. In general, this will involve an integral or sum over the strain at various space and time points. We impose the condition that the law be local, which is to say that the stress at a certain position depends only on the strain at that position. This amounts to excluding "action at a distance" forces such as might arise for example if the material were susceptible to electromagnetic fields which in turn were dependent on the mechanical state of the material. Furthermore, the Principle of Causality implies that the stress depends only on present and past values of the strain. We write

$$\sigma(x,t) = f(t;x)\varepsilon(x,t) + \int_{-\infty}^{t} dt' g(t,t';x)\varepsilon(x,t') \ . \tag{1.2.1}$$

Since the functions $f(t;x)$, $g(t,t';x)$ characterize the response of the material, their dependence upon x implies that the material is spatially inhomogeneous. For most of the present volume, a homogeneous material will be assumed. However, there are important problem classes for which inhomogeneity cannot be neglected, notably where it arises from space-dependent environmental effects, in particular temperature. This type of inhomogeneity is discussed in Sect. 1.7.

The space dependence of $f(t;x)$, $g(t,t';x)$ will not effect the considerations of this section and so explicit reference to it will be dropped. An alternative way of writing (1.2.1) is as follows:

$$\sigma(t) = G(t, -\infty)\varepsilon(-\infty) + \int_{-\infty}^{t} dt' G(t,t')\dot{\varepsilon}(t') \ , \qquad \text{where} \tag{1.2.2}$$

$$\frac{d}{dt'} G(t,t') = -g(t,t') \ , \qquad G(t,t) = f(t) \ . \tag{1.2.3}$$

If $g(t,t'), f(t)$ are given, it is always possible to choose $G(t,t')$ such that (1.2.3) is satisfied. This is clear if one observes that the first condition determines $G(t,t')$ only up to an arbitrary function of t, which can be chosen so that the second condition is satisfied.

Integrals over the history of strain (or stress) as occur in (1.2.1, 2) are sometimes referred to as hereditary integrals. Materials whose constitutive equations contain such hereditary integrals are described as having memory.

Observe that (1.2.3) implies that $G(t,t')$ must be differentiable with respect to its second argument. This is a smoothness requirement on the function characterizing the dependence of $\sigma(t)$ on the past history of $\dot{\varepsilon}(t)$.

In may cases, $\varepsilon(-\infty)$ will vanish, thus eliminating the first term in (1.2.2). If the strain is zero before a certain time t_1, then (1.2.1) becomes, with the aid of (1.2.3),

$$\sigma(t) = G(t,t)\varepsilon(t) - \int_{t_1}^{t} dt' \left[\frac{d}{dt'} G(t,t') \right] \varepsilon(t') \ . \tag{1.2.4}$$

Under these circumstances, (1.2.2) becomes

$$\sigma(t) = G(t,t_1)\varepsilon(t_1) + \int_{t_1}^{t} dt' G(t,t')\dot{\varepsilon}(t') \tag{1.2.5}$$

as may be deduced by partial integration of (1.2.4). Equations (1.2.4, 5) are equivalent and equally important forms of the same physical statement. Constitutive equations cast in the form of (1.2.5) are perhaps most commonly used in the literature. It has advantages, particularly where $\varepsilon(t)$ is constant over certain time intervals. Equation (1.2.4) however expresses stress as a linear functional of strain in a straightforward manner, essentially generalizing simple matrix multiplication. It has practical advantages and an adaptation of it will be used in much of the detailed work of later chapters.

If the hereditary integral is absent in (1.2.4), the relation reduces to Hooke's Law where the modulus is time-dependent. Note that the dimension of $G(t,t')$ is the same as the moduli in elastic theory, namely force per unit area.

The integral in (1.2.5) is really a special case (where $\varepsilon(t)$ is differentiable) of a Stieltjes integral. Gurtin and Sternberg (1962) base their rigorous formulation of Linear Viscoelasticity on constitutive equations which have this Stieltjes form. We adopt a convenient notation of theirs, and write (1.2.5) as

$$\sigma(t) = [G * d\varepsilon](t) \ , \tag{1.2.6}$$

which we term a Stieltjes product.

One can view G in (1.2.6) as a linear operator acting on ε, which may be thought of as a vector in an abstract space. The concrete realization of G, ε are the functions $G(t, t')$, $\varepsilon(t)$, the relationship between them being a generalization of that between a linear operator and matrix elements or vector and vector components.

1.2.2 The Operator Algebra

It is useful to adopt a somewhat more abstract viewpoint on operators of this type. Our treatment, however, falls far short of thoroughgoing rigour.

Consider the class of operators corresponding to all functions $G(t, t')$ which are differentiable with respect to t, t' and which obey the Causality condition

$$G(t, t') = 0 \; , \quad t < t' \; . \tag{1.2.7}$$

We impose a further condition that

$$G(t, t) \neq 0 \tag{1.2.8}$$

for all t, except where $G(t, t')$ is identically zero, for all t, t'. The functions $\varepsilon(t)$ corresponding to the vectors ε on which the operators act will be assumed to be zero for $t < t_1$.

Addition and multiplication of operators of this kind can be defined in a straightforward manner. The sum of two operators G_1, G_2 is that operator associated with $G_1(t, t') + G_2(t, t')$. It is easy to show that addition in this sense obeys all the usual rules. Similarly, multiplication of an operator by a constant may be defined in an obvious manner. The zero operator is that corresponding to $G(t, t')$ equal to zero for all t, t'.

Multiplication of operators is defined as the consecutive application of two operators. In other words, for all ε,

$$[(G_2 * dG_1) * d\varepsilon](t) = [G_2 * d(G_1 * d\varepsilon)](t) \; . \tag{1.2.9}$$

From this definition, it may be shown, with the aid of (1.2.4, 5), and an interchange in the order of integration, that

$$[G_2 * dG_1](t, t') = G_2(t, t) G_1(t, t') - \int_{t'}^{t} dt'' \left[\frac{d}{dt''} G_2 \, (t, t'') \right] G_1(t'', t') \tag{1.2.10a}$$

$$= G_2(t, t') G_1(t', t') + \int_{t'}^{t} dt'' G_2(t, t'') \frac{d}{dt''} G_1(t'', t') \; . \tag{1.2.10b}$$

Observe that in the second form, the derivative is with respect to the first rather than the second time argument, which is why differentiability with respect to both arguments is required.

Using essentially the same type of manipulation, one can show that this product is associative, i.e.

$$[G_3 * d(G_2 * dG_1)](t, t') = [(G_3 * dG_2) * dG_1](t, t') \; . \tag{1.2.11}$$

It is easy to show that it is distributive with respect to addition:

$$[G_3 * d(G_1 + G_2)](t, t') = [G_3 * dG_1](t, t') + [G_3 * dG_2](t, t') \ . \tag{1.2.12}$$

Finally, one can define a unit operator I, of the form

$$I(t, t') = H(t - t') \ , \tag{1.2.13}$$

where $H(t)$ is the Heaviside step function defined by (A 3.1.3 p). The following properties may be demonstrated:

$$[I * d\varepsilon](t) = \varepsilon(t), \quad [I * dG](t, t') = G(t, t') = [G * dI](t, t') \ . \tag{1.2.14}$$

It will now be shown that if $G_2 * dG_1$ is zero, then either G_1 or G_2 is zero. The proof requires a fundamental property of Volterra integral equations, discussed in Sect. A4.2. If $[G_2 * dG_1](t, t')$ is zero, then by definition, from (1.2.10a):

$$G_2(t, t) G_1(t, t') - \int_{t'}^{t} dt'' \left[\frac{d}{dt''} G_2(t, t'') \right] G_1(t'', t') = 0 \ . \tag{1.2.15}$$

Let us assume that G_2 is non-zero. It follows from (1.2.8) that $G_2(t, t)$ is non-zero for all t. Equation (1.2.15) is therefore a homogeneous Volterra equation, for $G_1(t, t')$ at fixed t'. Such equations have only the trivial zero solution, so that $G_1(t, t') = 0$. If G_1 is assumed to be non-zero, it can be shown similarly, with the aid of (1.2.10b) that G_2 must be zero.

Finally, a very important property for later use, is that every operator G has a unique inverse J in this class, such that

$$G * dJ = J * dG = I \ . \tag{1.2.16}$$

The requirement that J be the right inverse is given explicitly by

$$G(t, t) J(t, t') - \int_{t'}^{t} dt'' \left[\frac{d}{dt''} G(t, t'') \right] J(t'', t') = H(t - t') \ . \tag{1.2.17}$$

This is an inhomogeneous Volterra integral equation of the second kind for $J(t, t')$ at fixed t', which always has a unique non-zero solution. It gives the form of $J(t, t')$ for $t \geq t'$. For $t < t'$ we take it to be zero so that it obeys the Causality condition. Furthermore, from (1.2.17), it follows that for all t

$$G(t, t) J(t, t) = 1 \tag{1.2.18}$$

so that $J(t, t)$ cannot be zero anywhere. Therefore, J is an operator in the class of interest. To show that it is also a left inverse, we consider

$$K = (J * dG - I) * dJ = J * d(G * dJ) - J = 0. \tag{1.2.19}$$

It follows that one of the factors in the product is zero. Since J is not zero, one must have $J * dG = I$.

An immediate consequence of the existence of an inverse is that the constitutive equation (1.2.5) may be written as

$$\varepsilon(t) = [J * d\sigma](t) = J(t, t_1) \sigma(t_1) + \int_{t_1}^{t} dt' J(t, t') \dot{\sigma}(t') \tag{1.2.20}$$

or, in the form of (1.2.4),

$$\varepsilon(t) = J(t,t)\sigma(t) - \int_{t_1}^{t} dt' \left[\frac{d}{dt'} J(t,t')\right] \sigma(t') \tag{1.2.21}$$

where the function $J(t,t')$ is uniquely determined by $G(t,t')$ and vice versa. We refer to $G(t,t')$, $J(t,t')$ as the relaxation and creep functions, for physical reasons which will be discussed later in Sect. 1.4.

1.2.3 Alternative Notation

Equations (1.2.4), and (1.2.21) may be written as

$$\begin{aligned} \sigma(t) &= \int_{t_1}^{t} dt' \mu(t,t')\varepsilon(t') = [\mu*\varepsilon](t) \\ \varepsilon(t) &= \int_{t_1}^{t} dt' \gamma(t,t')\sigma(t') = [\gamma*\varepsilon](t) \ , \quad \text{where} \end{aligned} \tag{1.2.22}$$

$$\begin{aligned} \mu(t,t') &= G(t,t)\delta(t-t') - H(t-t') \frac{d}{dt'} G(t,t') \\ &= -\frac{d}{dt'} [H(t-t')G(t,t')] \\ \gamma(t,t') &= J(t,t)\delta(t-t') - H(t-t') \frac{d}{dt'} J(t,t') \\ &= -\frac{d}{dt'} [H(t-t')J(t,t')] \ , \end{aligned} \tag{1.2.23}$$

and $\delta(t)$ is the singular delta function, discussed in Sect. A3.1. The upper limits of the integrals in (1.2.22) are understood to be t^+, the limiting value from above, so that the full contribution of the delta functions are picked up. This compact notation is very useful for formal development and manipulation. It will be widely used in later chapters – generally in the context of non-aging materials, which are introduced below. It tends to be avoided in more rigorous mathematical treatments of viscoelastic theory, since rigorous use of the delta function involves some cumbersome mathematical theory. Note that $\mu(t,t')$, $\gamma(t,t')$ obey the Causality relations

$$\mu(t,t') = \gamma(t,t') = 0, \qquad t < t' \ . \tag{1.2.24}$$

Also, from (1.2.22), it can be shown that

$$\int_{t'}^{t} dt'' \mu(t,t'') \gamma(t'',t') = \int_{t'}^{t} dt'' \gamma(t,t'') \mu(t'',t') = \delta(t-t') \tag{1.2.25}$$

with the aid of an interchange in the order of integration. The first of these is just (1.2.17) in differentiated form.