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Manifolds, Tensor Analysis, and Applications

Second Edition

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Preface

The purpose of this book is to provide core material in nonlinear analysis for mathematicians, physicists, engineers, and mathematical biologists. The main goal is to provide a working knowledge of manifolds, dynamical systems, tensors, and differential forms. Some applications to Hamiltonian mechanics, fluid mechanics, electromagnetism, plasma dynamics and control theory are given in Chapter 8, using both invariant and index notation. The current edition of the book does not deal with Riemannian geometry in much detail, and it does not treat Lie groups, principal bundles, or Morse theory. Some of this is planned for a subsequent edition. Meanwhile, the authors will make available to interested readers supplementary chapters on Lie Groups and Differential Topology and invite comments on the book's contents and development.

Throughout the text supplementary topics are given, marked with the symbols ☞ and ✍. This device enables the reader to skip various topics without disturbing the main flow of the text. Some of these provide additional background material intended for completeness, to minimize the necessity of consulting too many outside references.

We treat finite and infinite-dimensional manifolds simultaneously. This is partly for efficiency of exposition. Without advanced applications, using manifolds of mappings, the study of infinite-dimensional manifolds can be hard to motivate. Chapter 8 gives a hint of these applications. In fact, some readers may wish to skip the infinite-dimensional case altogether. To aid in this we have separated into supplements some of the technical points peculiar to the infinite-dimensional case. Our own research interests lean toward physical applications, and the choice of topics is partly molded by what is useful for this kind of research. We have tried to be as sympathetic to our readers as possible by providing ample examples, exercises, and applications. When a computation in coordinates is easiest, we give it and do not hide things behind complicated invariant notation. On the other hand, index-free notation sometimes provides valuable geometric and computational insight so we have tried to simultaneously convey this flavor.

The prerequisites required are solid undergraduate courses in linear algebra and advanced calculus. At various points in the text contacts are made with other subjects, providing a good way for students to link this material with other courses. For example, Chapter 1 links with point-set topology, parts of Chapter 2 and 7 are connected with functional analysis, Section 4.3 relates to ordinary differential equations, Chapter 3 and Section 7.5 are linked to differential topology and algebraic topology, and Chapter 8 on applications is connected with applied mathematics, physics, and engineering.

This book is intended to be used in courses as well as for reference. The sections are, as far as possible, lesson sized, if the supplementary material is omitted. For some sections, like 2.5, 4.2, or 7.5, two lecture hours are required. A standard course for mathematics graduate students could omit Chapter 1 and the supplements entirely and do Chapters 2 through 7 in one semester with the possible exception of Section 7.4. The instructor could then assign certain supplements for reading and choose among the applications of Chapter 8 according to taste. A shorter course, or a course advanced undergraduates, probably should omit all supplements, spend about two lectures on Chapter 1 for reviewing background point set topology, and cover Chapters 2 through 7 with the exception of Sections 4.4, 7.4, 7.5 and all the material relevant to volume elements induced by metrics, the Hodge star, and codifferential operators in Sections 6.2, 6.4, 6.5, and 7.2. A more applications oriented course could skim Chapter 1, review without proofs the material of Chapter 2, and cover Chapters 3 to 8 omitting the supplementary material and Sections 7.4 and 7.5. For such a course the instructor should keep in mind that while Sections 8.1 and 8.2 use only elementary material, Section 8.3 relies heavily on the Hodge star and codifferential operators, and Section 8.4 consists primarily of applications of Frobenius' theorem dealt with in Section 4.4.

The notation in the book is as standard as conflicting usages in the literature allow. We have had to compromise among utility, clarity, clumsiness, and absolute precision. Some possible notations would have required too much interpretation on the part of the novice while others, while precise, would have been so dressed up in symbolic decorations that even an expert in the field would not recognize them.

In a subject as developed and extensive as this one, an accurate history and crediting of theorems is a monumental task, especially when so many results are folklore and reside in private notes. We have indicated some of the important credits where we know of them, but we did not undertake this task systematically. We hope our readers will inform us of these and other shortcomings of the book so that, if necessary, corrected printings will be possible. The reference list at the back of the book is confined to works actually cited in the text. These works are cited by author and year like this: deRham [1955].

During the preparation of the book, valuable advice was provided by Malcolm Adams, Morris Hirsch, Charles Pugh, Alan Weinstein, and graduate students in mathematics, physics and engineering at Berkeley, Santa Cruz and elsewhere. Our other teachers and collaborators from whom we learned the material and who inspired, directly and indirectly, various portions of the text are too numerous to mention individually, so we hereby thank them all collectively. We have taken the opportunity in this edition to correct some errors kindly pointed out by our readers and to rewrite numerous sections. This book was typeset on a Macintosh using Mathwriter (Cooke Publications Inc, Ithaca, N.Y.); we thank Connie Calica, Dotty Hollinger, Marnie MacElhiny and Esther Zack for their invaluable help with the typing.

We intend this book to be an evolving project. That is, we invite corrections and comments from our readers to be incorporated into future printings. We are currently preparing some supplementary chapters and plan to include a differential topology and Lie groups chapter in the next printing—space permitting. Meanwhile, if you wish to see these chapters, we will be happy to send them to you in exchange for your comments.

February, 1988

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Contents

Background Notation

The reader is assumed to be familiar with the usual notations of set theory such as $\in$, $\subset$, $\cup$, $\cap$ and with the concept of a mapping. If A and B are sets and if $f: A \to B$ is a mapping, we write $a \mapsto f(a)$ for the effect of the mapping on the element of $a \in A$; "iff" stands for "if and only if" (= "if" in definitions). Other notations we shall use without explanation include the following:

♦	end of an example or remark
■	end of a proof
▼	proof of a lemma is done, but the proof of the theorem goes on
$\mathbb{R}$, $\mathbb{C}$	real, complex numbers
$\mathbb{Z}$, $\mathbb{Q}$	integers, rational numbers
$A \times B$	Cartesian product
$\mathbb{R}^n$, $\mathbb{C}^n$	Euclidean n-space, complex n-space
$(x^1, \ldots, x^n) \in \mathbb{R}^n$	point in $\mathbb{R}^n$
$A \subset B$	set theoretic containment (means same as $A \subseteq B$)
$A \setminus B$	set theoretic difference
I or Id	identity map
$f^{-1}(B)$	inverse image of B under f
$\Gamma_f = \{(x, f(x)) \mid x \in \text{domain of } f\}$	graph of f
inf A	infimum (greatest lower bound) of the set $A \subset \mathbb{R}$
sup A	supremum (least upper bound) of $A \subset \mathbb{R}$
$e_1, \ldots, e_n$	basis of an n-dimensional vector space
ker T, range T	kernel and range of a linear transformation T
$D_r(m)$	open ball about m of radius r
$B_r(m)$	closed ball of radius r (also denoted $\overline{D}_r(m)$).

Chapter 1
Topology

The purpose of this chapter is to introduce just enough topology for later requirements. It is assumed that the reader has had a course in advanced calculus and so is acquainted with open, closed, compact, and connected sets in Euclidean space (see for example Marsden [1974a] and Rudin [1976]). If this background is weak, the reader may find the pace of this chapter too fast. If the background is under control, the chapter should serve to collect, review, and solidify concepts in a more general context. Readers already familiar with point set topology can safely skip this chapter.

A key concept in manifold theory is that of a differentiable map between manifolds. However, manifolds are also topological spaces and differentiable maps are continuous. Topology is the study of continuity in a general context; it is therefore appropriate to begin with it. Topology often involves interesting excursions into pathological spaces and exotic theorems. Such excursions are deliberately minimized here. The examples will be ones most relevant to later developments, and the main thrust will be to obtain a working knowledge of continuity, connectedness, and compactness.

We shall take for granted the usual logical structure of analysis without much comment, except to recall one of the basic axioms that is in common use and an equivalent result. These will be used occasionally in the text.

Axiom of choice *If* $\mathcal{S}$ *is a collection of nonempty sets, then there is a function*
$\chi : \mathcal{S} \to \bigcup_{S \in \mathcal{S}} S$ *such that* $\chi(S) \in S$ *for every* $S \in \mathcal{S}$.

The function χ chooses one element from each $S \in \mathcal{S}$ and is called a ***choice function.*** Even though this statement seems self-evident, it has been shown to be equivalent to a number of nontrivial statements, using other axioms of set theory. To discuss them, we need a few definitions. An ***order*** on a set A is a binary relation, usually denoted by "≤" satisfying the following conditions:

$a \leq a$	(***reflexivity***)
$a \leq b$ and $b \leq a$ implies $a = b$	(***antisymmetry***), and
$a \leq b$ and $b \leq c$ implies $a \leq c$	(***transitivity***).

An ordered set A is called a ***chain*** if for every $a, b \in A$, $a \neq b$ we have $a \leq b$ or $b \leq a$. The set A is said to be ***well ordered*** if it is a chain and every nonempty subset B has a first element; i.e., there exists an element $b \in B$ such that $b \leq x$ for all $x \in B$. An ***upper bound*** $u \in A$ of a chain $C \subset A$ is an element for which $c \leq u$ for all $c \in C$. A ***maximal element*** m of an ordered set A is an element for which there is no other $a \in A$ such that $m \leq a$, $a \neq m$; in other words $x \leq m$ for all $x \in A$ that are comparable to m. We state the following without proof.

Theorem *Given other axioms of set theory, the following statements are equivalent:*

(i) *The axiom of choice.*

(ii) **Product Axiom** *If* $\{A_i\}_{i\in I}$ *is a collection of nonempty sets then the product space* $\prod_{i\in I} A_i = \{(x_i) \mid x_i \in A_i\}$ *is nonempty.*

(iii) **Zermelo's Theorem** *Any set can be well ordered.*

(iv) **Zorn's Theorem** *If* A *is an ordered set for which every chain has an upper bound (i.e.,* A *is* ***inductively ordered****), then* A *has at least one maximal element.*

§1.1 *Topological Spaces*

Abstracting ideas about open sets in $\mathbb{R}^n$ leads to the notion of a topological space.

1.1.1 Definition *A* ***topological space*** *is a set* S *together with a collection* $\mathcal{O}$ *of subsets called* ***open sets*** *such that*

T1 $\varnothing \in \mathcal{O}$ *and* $S \in \mathcal{O}$;

T2 *if* $U_1, U_2 \in \mathcal{O}$, *then* $U_1 \cap U_2 \in \mathcal{O}$;

T3 *the union of any collection of open sets is open.*

A basic example is the real line. We choose $S = \mathbb{R}$, with $\mathcal{O}$ consisting of all sets that are unions of open intervals. As exceptional cases, the empty set $\varnothing \in \mathcal{O}$ and $\mathbb{R}$ itself belong to $\mathcal{O}$. Thus **T1** holds. For **T2**, let U_1 and $U_2 \in \mathcal{O}$; to show that $U_1 \cap U_2 \in \mathcal{O}$, we can suppose that $U_1 \cap U_2 \neq \varnothing$. If $x \in U_1 \cap U_2$, then x lies in an open interval $]a_1, b_1[\subset U_1$ and also in the interval $]a_2, b_2[\subset U_2$. We can write $]a_1, b_1[\cap]a_2, b_2[=]a, b[$ where $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2))$. Thus $x \in]a, b[\subset U_1 \cap U_2$. Hence $U_1 \cap U_2$ is the union of such intervals, so is open. Finally, **T3** is clear by definition.

Similarly, $\mathbb{R}^n$ may be topologized by declaring a set to be open if it is a union of open rectangles. An argument similar to the one just given for $\mathbb{R}$ shows that this is a topology, called the ***standard topology*** on $\mathbb{R}^n$.

The ***trivial topology*** on a set S consists of $\mathcal{O} = \{\varnothing, S\}$. The ***discrete topology*** on S is defined by $\mathcal{O} = \{A \mid A \subset S\}$; i.e., $\mathcal{O}$ consists of all subsets of S.

Topological spaces are specified by a pair $(S, \mathcal{O})$; we shall, however, simply write S if there is no danger of confusion.

1.1.2 Definition *Let* S *be a topological space. A set* $A \subset S$ *will be called* ***closed*** *if its complement* $S \setminus A$ *is open. The collection of closed sets is denoted* $\mathcal{C}$.

For example, the closed interval $[0, 1] \subset \mathbb{R}$ is closed as it is the complement of the open set $]-\infty, 0[\cup]1, \infty[$.

1.1.3 Proposition *The closed sets in a topological space satisfy:*

C1 $\varnothing \in \mathcal{C}$ *and* $S \in \mathcal{C}$;

C2 *if* $A_1, A_2 \in \mathcal{C}$ *then* $A_1 \cup A_2 \in \mathcal{C}$;

C3 *the intersection of any collection of closed sets is closed.*

Proof **C1** follows from **T1** since $\varnothing = S \setminus S$, $S = S \setminus \varnothing$. The relations

$$S \setminus (A_1 \cup A_2) = (S \setminus A_2) \cap (S \setminus A_2) \quad \text{and} \quad S \setminus \left(\bigcap_{i \in I} B_i\right) = \bigcup_{i \in I} (S \setminus B_i)$$

for $\{B_i\}_{i \in I}$ a family of closed sets show that **C2**, **C3** are equivalent to **T2**, **T3**, respectively. ■

Closed rectangles in $\mathbb{R}^n$ are closed sets, as are closed balls, one-point sets, and spheres. Not every set is either open or closed. For example, the interval $[0, 1[$ is neither an open nor a closed set. In a discrete topology on S any set $A \subset S$ is both open and closed, whereas in the trivial topology any $A \neq \varnothing$ or S is neither.

Closed sets can be used to introduce a topology just as well as open ones. Thus, if $\mathcal{C}$ is a collection satisfying **C1-C3** and $\mathcal{O}$ consists of the complements of sets in $\mathcal{C}$, then $\mathcal{O}$ satisfies **T1-T3.**

1.1.4 Definition *An* ***open neighborhood*** *of a point* u *in a topological space* S *is an open set* U *such that* $u \in U$. *Similarly, for a subset* A *of* S, U *is an open neighborhood of* A *if* U *is open and* $A \subset U$. *A* ***neighborhood*** *of a point (or a subset) is a set containing some open neighborhood of the point (or subset).*

Examples of neighborhoods of $x \in \mathbb{R}$ are $]x - 1, x + 3]$, $]x - \varepsilon, x + \varepsilon[$ for any $\varepsilon > 0$, and $\mathbb{R}$ itself; only the last two are open neighborhoods. The set $[x, x + 2[$ contains the point x but is not one of its neighborhoods. In the trivial topology on a set S, there is only one neighborhood of any point, namely S itself. In the discrete topology any subset containing p is a neighborhood of the point $p \in S$, since $\{p\}$ is an open set.

1.1.5 Definition *A topological space is called* ***first countable*** *if for each* $u \in S$ *there is a sequence* $\{U_1, U_2, \ldots\} = \{U_n\}$ *of neighborhoods of* u *such that for any neighborhood* U *of* u, *there is an integer* n *such that* $U_n \subset U$. *A subset* $\mathcal{B}$ *of* $\mathcal{O}$ *is called a* ***basis*** *for the topology, if each open set is a union of elements in* $\mathcal{B}$. *The topology is called* ***second countable*** *if it has a countable basis.*

Most topological spaces of interest to us will be second countable. For example $\mathbb{R}^n$ is second countable since it has the countable basis formed by rectangles with rational side length and centered at points all of whose coordinates are rational. Clearly every second-countable space is also first countable, but the converse is false. For example if S is an infinite noncountable set, the discrete topology is not second countable, but S is first countable, since $\{p\}$ is a neighborhood of $p \in S$. The trivial topology on S is second countable (see Exercises **1.1I, 1.1J** for more interesting counter-examples).

1.1.6 Lindelöf's Lemma *Every covering of a set* A *in a second countable space* S *by a family of open sets* U_a *(that is* $\bigcup_a U_a \supset A$*) contains a countable subcollection also covering* A.

Proof Let $\mathcal{B} = \{B_n\}$ be a countable basis for the topology of S. For each $p \in A$ there are indices n and α such that $p \in B_n \subset U_\alpha$. Let $\mathcal{B}' = \{B_n \mid \text{there exists an } \alpha \text{ such that } B_n \subset U_\alpha\}$. Now let $U_{\alpha(n)}$ be one of the U_α that includes the element B_n of $\mathcal{B}'$. Since $\mathcal{B}'$ is a covering of A, the countable collection $\{U_{\alpha(n)}\}$ covers A. ■

1.1.7 Definition *Let* S *be a topological space and* $A \subset S$. *The* ***closure*** *of* A, *denoted* cl(A) *is the intersection of all closed sets containing* A. *The* ***interior*** *of* A, *denoted* int(A) *is the union of all open sets contained in* A. *The* ***boundary*** *of* A, *denoted* bd(A) *is defined by*

$$\text{bd}(A) = \text{cl}(A) \cap \text{cl}(S \setminus A).$$

By **C3**, cl(A) is closed and by **T3**, int(A) is open. Note that as bd(A) is the intersection of closed sets, bd(A) is closed, and bd(A) = bd(S \ A)

On $\mathbb{R}$, for example, cl([0, 1[) = [0, 1], int([0, 1[) =]0, 1[, and bd([0, 1[) = {0, 1}. The reader is assumed to be familiar with examples of this type from advanced calculus.

1.1.8 Definitions *A subset* A *of* S *is called* ***dense in*** S *if* cl(A) = S, *and is called* ***nowhere dense*** *if* S \ cl(A) *is dense in* S. *The space* S *is called* ***separable*** *if it has a countable dense subset. A point in* S *is called an* ***accumulation point*** *of the set* A *if each of its neighborhoods contains a point of* A *other than itself. The set of accumulation points of* A *is called the* ***derived set*** *of* A *and is denoted by* der(A). *A point of* A *is said to be* ***isolated*** *if it has a neighborhood in* S *containing no other points of* A *than itself.*

The set A = [0,1[∪ {2} in $\mathbb{R}$ has the element 2 as its only isolated point, its interior is int(A) =]0, 1[, cl(A) = [0, 1] ∪ {2} and der(A) = [0, 1]. In the discrete topology on a set S, int{p} = cl{p} = {p}, for any $p \in S$.

Since the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$ and is countable, $\mathbb{R}$ is separable. Similarly $\mathbb{R}^n$ is separable. A set S with the trivial topology is separable since cl{p} = S for any $p \in S$. But $S = \mathbb{R}$ with the discrete topology is not separable since cl(A) = A for any $A \subset S$. Any second-countable space is separable, but the converse is false; see Exercises **1.1I, 1.1J**.

1.1.9 Proposition *Let* S *be a topological space and* $A \subset S$. *Then*

(i) $u \in cl(A)$ *iff for every neighborhood* U *of* u, $U \cap A \neq \emptyset$;

(ii) $u \in int(A)$ *iff there is a neighborhood* U *of* u *such that* $U \subset A$;

(iii) $u \in bd(A)$ *iff for every neighborhood* U *of* u, $U \cap A \neq \emptyset$ *and* $U \cap (S \setminus A) \neq \emptyset$.

Proof (i) $u \notin cl(A)$ iff there exists a closed set $C \supset A$ such that $u \notin C$. But this is equivalent to the existence of a neighborhood of u not intersecting A, namely $S \setminus C$. (ii) and (iii) are proved in a similar way. ■

1.1.10 Proposition *Let* A, B *and* A_i, $i \in I$ *be subsets of* S.

(i) $A \subset B$ *implies* $int(A) \subset int(B)$, $cl(A) \subset cl(B)$, *and* $der(A) \subset der(B)$;

(ii) $S \setminus cl(A) = int(S \setminus A)$, $S \setminus int(A) = cl(S \setminus A)$, *and* $cl(A) = A \cup der(A)$;

(iii) $cl(\emptyset) = int(\emptyset) = \emptyset$, $cl(S) = int(S) = S$, $cl(cl(A)) = cl(A)$ *and* $int(int(A)) = int(A)$;

(iv) $cl(A \cup B) = cl(A) \cup cl(B)$, $der(A \cup B) = der(A) \cup der(B)$, $int(A \cup B) \supset int(A) \cup int(B)$;

(v) $cl(A \cap B) \subset cl(A) \cap cl(B)$, $der(A \cap B) \subset der(A) \cap der(B)$, $int(A \cap B) = int(A) \cap int(B)$;

(vi) $cl(\bigcup_{i \in I} A_i) \supset \bigcup_{i \in I} cl(A_i)$, $cl(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} cl(A_i)$,
$int(\bigcup_{i \in I} A_i) \supset \bigcup_{i \in I} int(A_i)$, $int(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} int(A_i)$.

Proof (i), (ii), and (iii) are consequences of the definition and of Proposition **1.1.9.** Since for each $i \in I$, $A_i \subset \bigcup_{i \in I} A_i$, by (i) $cl(A_i) \subset cl(\bigcup_{i \in I} A_i)$ and hence $\bigcup_{i \in I} cl(A_i) \subset cl(\bigcup_{i \in I} A_i)$. Similarly, since $\bigcap_{i \in I} A_i \subset A_i \subset cl(A_i)$ for each $i \in I$, it follows that $\bigcap_{i \in I} cl(A_i)$ is a subset of the closet set $\bigcap_{i \in I} cl(A_i)$; thus by (i) $cl(\bigcap_{i \in I} A_i) \subset cl(\bigcap_{i \in I} cl(A_i)) = \bigcap_{i \in I} (cl(A_i))$.The other formulas of (vi) follow from these and (ii). This also proves all the other formulas in (iv) and (v) except the ones with equalities. Since $cl(A) \cup cl(B)$ is closed by **C2** and $A \cup B \subset cl(A) \cup cl(B)$, it follows by (i) that $cl(A \cup B) \subset cl(A) \cup cl(B)$ and hence equality by (vi). The formula $int(A \cap B) = int(A) \cap int(B)$ is a corollary of the previous formula via (ii). ■

The inclusions in the above proposition can be strict. For example, if we let $A =]0,1[$ and $B = [1, 2[$, then one finds $cl(A) = der(A) = [0, 1]$, $cl(B) = der(B) = [1, 2]$, $int(A) =]0, 1[$, $int(B) =]1, 2[$, $A \cup B =]0, 2[$, and $A \cap B = \emptyset$, and therefore $int(A) \cup int(B) =]0,1[\cup]1, 2[\neq]0, 2[= int(A \cup B)$, and $cl(A \cap B) = \emptyset \neq \{1\} = cl(A) \cap cl(B)$. Let $A_n =]-1/n, 1/n[$, $n = 1, 2, \ldots$; then $\bigcap_{n \geq 1} A_n = \{0\}$, $int(A_n) = A_n$ for all n, and $int(\bigcap_{n \geq 1} A_n) = \emptyset \neq \{0\} = \bigcap_{n \geq 1} int(A_n)$. Dualizing this via (ii) gives $\bigcup_{n \geq 1} cl(\mathbb{R} \setminus A_n) = \mathbb{R} \setminus \{0\} \neq \mathbb{R} = cl(\bigcup_{n \geq 1} (\mathbb{R} \setminus A_n))$. If $A \subset B$, there is, in general, no relation between the sets bd(A) and bd(B). For example, if $A = [0, 1]$ and $B = [0, 2]$, $A \subset B$, yet we have $bd(A) = \{0, 1\}$ and $bd(B) = \{0, 2\}$.

1.1.11 Definition *Let* S *be a topological space and* $\{u_n\}$ *a sequence of points in* S. *The sequence is said to* ***converge*** *if there is a point* $u \in S$ *such that for every neighborhood* U *of* u, *there is an* N *such that* $n \geq N$ *implies* $u_n \in U$. *We say that* u_n ***converges to*** u, *or* u *is a* ***limit point*** *of* $\{u_n\}$.

For example, the sequence $\{1/n\}$ in $\mathbb{R}$ converges to 0. It is obvious that limit points of sequences u_n of distinct points are accumulation points of the set $\{u_n\}$. In a first countable topological space any accumulation point of a set A is a limit of a sequence of elements of A. Indeed, if $\{U_n\}$ denotes the countable collection of neighborhoods of $a \in \mathrm{der}(A)$ given by definition **1.1.5**, then choosing for each n an element $a_n \in U_n \cap A$ such that $a_n \neq a$, we see that $\{a_n\}$ converges to a. We have proved the following.

1.1.12 Proposition *Let* S *be a first-countable space and* $A \subset S$. *Then* $u \in \mathrm{cl}(A)$ *iff there is a sequence of points of* A *that converges to* u *(in the topology of* S).

It should be noted that a sequence can be divergent and still have accumulation points. For example $\{2, 0, 3/2, -1/2, 4/3, -2/3, \ldots\}$ does not converge but has both 1 and -1 as accumulation points. In arbitrary topological spaces, limit points of sequences are in general *not* unique. For example, in the trivial topology of S any sequence converges to all points of S. In order to avoid such situations several ***separation axioms*** have been introduced, of which the three most important ones will be mentioned.

1.1.13 Definition *A topological space* S *is called* ***Hausdorff*** *if each two distinct points have disjoint neighborhoods (that is, with empty intersection). The space* S *is called* ***regular*** *if it is Hausdorff and if each closed set and point not in this set have disjoint neighborhoods. Similarly,* S *is called* ***normal*** *if it is Hausdorff and if each two disjoint closed sets have disjoint neighborhoods.*

Most standard spaces in analysis are normal. The discrete topology on any set is normal, but the trivial topology is not even Hausdorff. It turns out that "Hausdorff" is the necessary and sufficient condition for uniqueness of limit points of sequences in first countable spaces (see Exercise **1.1E**). Since in Hausdorff space single points are closed (Exercise **1.1F**), we have the implications: normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff. Counterexamples for each of the converses of these implications are given in Exercises **1.1I** and **1.1J**.

1.1.14 Proposition *A regular second-countable space is normal.*

Proof Let A and B be two disjoint closed sets in S. By regularity, for every point $p \in A$ there are disjoint open neighborhoods U_p of p and U_B of B. Hence $\mathrm{cl}(U_p) \cap B = \varnothing$. Since $\{U_p \mid p \in A\}$ is an open covering of A, by the Lindelöf lemma (**1.1.6**), there is a countable collection $\{U_k \mid k = 1, 2, \ldots\}$ covering A. Thus $\bigcup_{k\geq 1} U_k \supset A$ and $\mathrm{cl}(U_k) \cap B = \varnothing$.

Similarly, find a family $\{V_k\}$ such that $\bigcup_{k\geq 0} V_k \supset B$ and $\mathrm{cl}(V_k) \cap A = \varnothing$. Then the sets $G_{n+1} = U_{n+1} \setminus \bigcup_{k = 0,1,\ldots,n} \mathrm{cl}(V_k)$, $H_n = V_n \setminus \bigcup_{k = 0,1,\ldots,n} \mathrm{cl}(U_k)$, $G_0 = U_0$, are open and $G = \bigcup_{n\geq 0} G_n \supset A$, $H = \bigcup_{n\geq 0} H_n \supset B$ are also open and disjoint. ■

In the remainder of this book Euclidean n-space $\mathbb{R}^n$ will be understood to have the standard topology (unless explicitly stated to the contrary).

Exercises

1.1A Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x < 1 \text{ and } y^2 + z^2 \le 1\}$. Find int(A).

1.1B Show that any finite set in $\mathbb{R}^n$ is closed.

1.1C Find the closure of $\{1/n \mid n = 1,2,\ldots\}$ in $\mathbb{R}$.

1.1D Let $A \subset \mathbb{R}$. Show that $\sup(A) \in \mathrm{cl}(A)$ where $\sup(A)$ is the supremum (l.u.b.) of A.

1.1E Show that a first countable space is Hausdorff iff all sequences have at most one limit point.

1.1F (i) Prove that in a Hausdorff space, single points are closed.

(ii) Prove that a topological space is Hausdorff iff the intersection of all closed neighborhoods of a point equals the point itself.

1.1G Show that in a Hausdorff space S the following are equivalent; (i) S is regular; (ii) for every point $p \in S$ and any of its neighborhoods U, there exists a closed neighborhood V of p such that $V \subset U$; (iii) for any closed set A, the intersection of all of the closed neighborhoods of A equals A.

1.1H (i) Show that if $\mathcal{V}(p)$ denotes the set of all neighborhoods of $p \in S$, then the following are satisfied:

V1 if $A \supset U$ and $U \in \mathcal{V}(p)$, then $A \in \mathcal{V}(p)$;

V2 every finite intersection of elements in $\mathcal{V}(p)$ is an element of $\mathcal{V}(p)$;

V3 p belongs to all elements of $\mathcal{V}(p)$;

V4 if $V \in \mathcal{V}(p)$ then there is a set $U \in \mathcal{V}(p)$, $U \subset V$ such that for all $q \in U$, $U \in \mathcal{V}(q)$.

(ii) If for each $p \in S$ there is a family $\mathcal{V}(p)$ of subsets of S satisfying **V1-V4**, prove that there is a unique topology $\mathcal{O}$ on S such that for each $p \in S$, the family $\mathcal{V}(p)$ is the set of neighborhoods of p in the topology $\mathcal{O}$. (*Hint:* Prove uniqueness first and then define elements of $\mathcal{O}$ as being subsets $A \subset S$ satisfying: for each $p \in A$, we have $A \in \mathcal{V}(p)$.)

1.1I Let $S = \{p = (x,y) \in \mathbb{R}^2 \mid y \ge 0\}$ and let $D_\varepsilon(p) = \{q \mid \| q - p \| < \varepsilon\}$ denote the usual ε-disk about p in the plane $\mathbb{R}^2$. Define

$$B_\varepsilon(p) = \begin{cases} D_\varepsilon(p) \cap S, & \text{if } p = (x, y) \text{ with } y > 0 \\ \{(x, y) \in D_\varepsilon(p) \mid y > 0\} \cup \{p\}, & \text{if } p = (x, 0). \end{cases}$$

Prove the following:

(i) $\mathcal{V}(p) = \{U \subset S \mid \text{there exists } B_\varepsilon(p) \subset U\}$ satisfies **V1-V4** of Exercise **1.1H**. Thus S becomes a toplogical space.

(ii) S is first countable.

(iii) S is Hausdorff.

(iv) S is separable. (*Hint* : The set $\{(x, y) \in S \mid x, y \in \mathbb{Q}, y > 0\}$ is dense in S.)

(v) S is not second countable (*Hint* : Assume the contrary and get a contradiction by looking at the points $(x, 0)$ of S.)

(vi) S is not regular. (*Hint:* Try to separate the point $(x_0, 0)$ from the set $\{(x, 0) \mid x \in \mathbb{R}\} \setminus \{(x_0, 0)\}$.)

1.1J With the same notations as in the preceding exercise, except changing $B_\varepsilon(p)$ to

$$B_\varepsilon(p) = \begin{cases} D_\varepsilon(p) \cap S, \text{ if } p = (x, y) \text{ with } y > 0 \\ \\ \{(x, y) \in D_\varepsilon(p) \mid y > 0\} \cup \{p\}, \text{ if } p = (x, 0), \end{cases}$$

show that (i)-(v) of **1.1I** remain valid and that

(vi) S is regular ; (*Hint:* Use Exercise **1.1G**.)

(vii) S is not normal. (*Hint* : Try to separate the set $\{(x, 0) \mid x \in \mathbb{Q}\}$ from the set $\{(x, 0) \mid x \in \mathbb{R} \setminus \mathbb{Q}\}$.)

1.1K Prove the following properties of the boundary operation and show by example that each inclusion cannot be replaced by equality.

Bd1 $\mathrm{bd}(A) = \mathrm{bd}(S \setminus A)$;

Bd2 $\mathrm{bd}(\mathrm{bd}(A) \subset \mathrm{bd}(A)$;

Bd3 $\mathrm{bd}(A \cup B) \subset \mathrm{bd}(A) \cup \mathrm{bd}(B) \subset \mathrm{bd}(A \cup B) \cup A \cup B$;

Bd3 $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) = \mathrm{bd}(\mathrm{bd}(A))$.

Properties **Bd1-Bd4** may be used to characterize the topology.

1.1L Let p be a polynomial in n variables $z_1, \ldots, z_n$ with complex coefficients. Show that $p^{-1}(0)$ has open dense complement. (*Hint:* If p vanishes on an open set of $\mathbb{C}^n$, then all its derivatives also vanish and hence all its coefficients are zero.)

1.1M Show that a subset $\mathcal{B}$ of $\mathcal{O}$ is a basis for the topology of S if and only if the following three conditions hold:

B1 $\varnothing \in \mathcal{B}$;

B2 $\bigcup_{B \in \mathcal{B}} B = S$;

B3 if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$.