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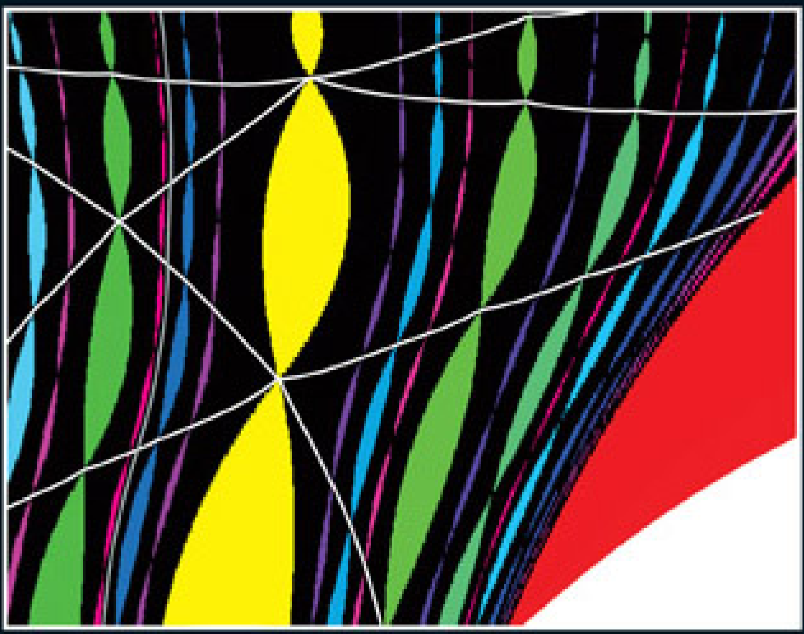
BIFURCATIONS IN PIECEWISE-SMOOTH CONTINUOUS SYSTEMS

David John Warwick Simpson



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Preface

Many real-world systems involve a discontinuity or sudden change, such as impacting in mechanical systems and switching in electrical circuits. Smooth dynamical systems generally do not provide ideal mathematical models for such situations. It becomes necessary to incorporate a nonsmooth component into the model. Often this yields a piecewise-smooth system.

Studies of piecewise-smooth systems prior to about twenty years ago are quite rare. Perhaps this is because the 1970's and 80's saw significant advances in the theory of smooth dynamical systems, such as an understanding of chaos. Nowadays smooth dynamical systems theory has a firm footing (though many open questions remain, e.g. Hilbert's 16^{th} problem) and piecewise-smooth systems are a popular topic of research. The advancement of piecewise-smooth theory has been led most notably by the Bristol school which has recently produced a unique and important textbook in this area [di Bernardo *et al.* (2008a)].

Although numerically computed solutions of a mathematical model at particular parameter values can give some understanding of the dynamics associated with the model, a more complete understanding relies on determining parameter values at which the qualitative features of these solutions change in some fundamental way, i.e. *bifurcations*. Of particular interest in this book is the nature of those bifurcations that are unique to piecewise-smooth systems. For instance in a piecewise-smooth system an attracting solution may lose stability and undergo an instantaneous transition to chaos. This had been observed in DC/DC converters for some time, see for instance [Fossas and Olivar (1996)], but explained only recently by nonsmooth bifurcation theory [di Bernardo *et al.* (1998b); Yuan *et al.* (1998)]. Unlike period-doubling cascades, period-adding sequences (which may occur in smooth systems) are not completely understood. However recent studies have explained such sequences in one-dimensional, piecewise-smooth maps [Avrutin *et al.* (2007); Halse *et al.* (2003); Kawczyński and Strizhak (2000)]. Also, resonance tongues (or mode-locking regions) that display a curious lens-chain structure have been observed in models of DC/DC converters [Zhusubaliyev and Mosekilde (2003)] and in a trade cycle model [Sushko and Gardini (2006)]. A novel description of the dynamics near points of zero width of such tongues is described here in Chapter 6.

Bifurcations unique to piecewise-smooth systems differ in many fundamental aspects to those in smooth systems. Often the dynamical behavior local to a bifurcation of a piecewise-smooth system is determined by only linear terms of an appropriate series expansion. This yields two immediate consequences. First, if nonlinear terms are put aside, the resulting approximation is piecewise-linear and relatively easy to study. This is because solutions of linear systems may be determined explicitly. In fact some researchers choose to study piecewise-linear approximations of smooth, nonlinear systems, for example [Bergami *et al.* (2006)]. Second, invariant sets that are created at such a bifurcation generically grow in size or separate linearly with respect to parameter values. This is different to the behavior of familiar smooth bifurcations, such as the Andronov-Hopf bifurcation [Kuznetsov (2004); Marsden and McCracken (1976)] at which a periodic orbit is created that grows in size as the square root of the bifurcation parameter.

An important characteristic of any bifurcation is its *codimension* - the number of parameters that need to be varied in order for the bifurcation to occur. In an arbitrary parameter space of a dimension equal to the codimension of the bifurcation, an *unfolding* of a bifurcation is a description of all dynamical behavior that may generically occur near the bifurcation. A large component of this book is the unfolding of codimension-two bifurcations. The *normal form* of a local bifurcation is a simple system that exhibits all aspects of the unfolding. For smooth systems the concept of dimension reduction via center manifold analysis allows bifurcations in systems of any number of dimensions to be transformed to their normal forms. This technique is crucial to understanding complex dynamical systems of high dimension. However in piecewise-smooth systems a lack of differentiability often disallows dimension reduction. Consequently little is known about bifurcations in piecewise-smooth systems of arbitrary dimension.

This book is essentially an updated and revised version of my PhD thesis [Simpson (2008)]. The book concerns piecewise-smooth systems that are continuous and autonomous. Both continuous-time systems (ODEs) and discrete-time systems (maps) are investigated. The smooth components of the systems will be assumed to contain no singularities that affect the dynamical behavior of interest (in particular, maps with a square-root or 3/2-type singularity [di Bernardo *et al.* (2001b)] are not considered). A brief overview of each chapter is now given.

Chapter 1 provides a background. Section 1.1 discusses applications; the remaining sections derive the piecewise equations used in later chapters and summarize key results of previous researchers. A variety of terminology used below and throughout this book is introduced in this chapter.

The next three chapters investigate piecewise-smooth, continuous ODE systems. Chapter 2 looks at discontinuous bifurcations in planar systems. Section 2.1 discusses periodic orbits. Section 2.2 studies a discontinuous analogue of a Hopf bifurcation. The main result of this section, Theorem 2.1, details the creation of a periodic orbit at this bifurcation in a general setting. A practical summary is given in Sec. 2.3 where it is shown that there are exactly four distinct discontinuous bifurcations in planar systems.

Chapter 3 presents unfoldings of three different codimension-two, discontinuous bifurcations. In the order given they are the simultaneous occurrence of a (smooth) saddle-node bifurcation and a discontinuous bifurcation, the simultaneous occurrence of a Hopf bifurcation and a discontinuous bifurcation, and a discontinuous Hopf bifurcation of indeterminable criticality. The first unfolding is accomplished for a piecewise-smooth, continuous ODE system of arbitrary dimension. The second scenario is completely determined for a two-dimensional system, then a partial result is obtained in N dimensions. The third unfolding is carried out in two dimensions.

Chapter 4 applies the theoretical results of the previous two chapters to a real physical system. The system studied is an eight-dimensional, cybernetic model of the continuous cultivation of *Saccharomyces cerevisiae* (a common yeast) formulated by Jones & Kompala [Jones and Kompala (1999)]. As detailed in Sec. 4.1, the model is piecewise-smooth, continuous, due to the key model assumption that the yeast switches between metabolic pathways in a manner that maximizes its growth rate. Nonsmoothness and other basic properties are discussed in Sec. 4.2. A bifurcation analysis is presented in Sec. 4.3. Here a variety of codimension-two, discontinuous bifurcations are found. These are shown to be in full agreement with the results of Chapter 3. Stable oscillations which arise via Hopf and discontinuous Hopf-like bifurcations are described in Sec. 4.4. These correspond to spontaneous oscillations observed in experiments. By a detailed inspection of time series plots, an explanation as to the cause of the oscillations is determined.

The next three chapters study discrete-time systems. Chapter 5 analyzes codimension-two, border-collision bifurcations. It is convenient that the piecewise-smooth ODE systems that describe discontinuous bifurcations, have the same form as the piecewise-smooth maps describing border-collision bifurcations. Much of the basic analysis of the continuous-time case may be carried over to the discrete-time case. This is most evident for the codimension-two, simultaneous occurrence of a saddle-node bifurcation and a border-collision bifurcation detailed in Sec. 5.1. Sec. 5.2 studies the simultaneous occurrence of a period-doubling bifurcation and a border-collision bifurcation. A complete description is obtained in one dimension; a partial result is constructed for higher dimensions.

Chapter 6 looks at periodic solutions of piecewise-smooth, continuous maps near border-collision bifurcations at a smooth switching manifold. Symbolic dynamics enable periodic solutions to be identified by symbol sequences that give their itinerary relative to the switching manifold. As a result, any periodic solution may be expressed as the solution to a linear system which is constructed in manner determined by the associated symbol sequence. Two important matrices appear in the given formulation of the linear system: the *stability matrix* and the *border-collision matrix*. Bifurcations of periodic solutions and thus resonance tongue boundaries are described in terms of the multipliers of the stability matrix and the singularity of the border-collision matrix. A new class of symbol sequences, called “rotational symbol sequences” is introduced in Sec. 6.4. Such symbol sequences correspond to periodic solutions that lie on an invariant circle. A reason for the success of the symbolic approach is that no assumptions need to be made about the existence of invariant circles. Algebraic properties of rotational symbol sequences are used to explain the lens-chain geometry of resonance tongues observed in two-parameter bifurcation diagrams (bifurcation sets). Points where adjacent lenses connect, i.e. where the resonance tongues have zero width, are termed “shrinking points”. A rigorous unfolding of shrinking points is performed in the final section.

Chapter 7 studies border-collision bifurcations in a planar, piecewise-smooth, continuous map for which the multipliers associated with the fixed point are complex and “jump” from inside to outside the unit circle at the bifurcation. These bifurcations are shown to exhibit a wide variety of Neimark-Sacker-like and nonsmooth behavior. An invariant circle may be created at the bifurcation. Mode-locking on the invariant circle corresponds to a point in a resonance tongue. The theoretical results of the previous chapter are compared with numerically computed lens-chain-shaped resonance tongues. Additional complex phenomena are also discussed. A curve of shrinking points is seen to be a boundary between chaotic and non-chaotic dynamics. Periodic solutions with non-rotational symbol sequences are found to belong to resonance tongues that do not display a lens-chain structure. Unlike at a generic border-collision bifurcation in a one-dimensional, piecewise-smooth, continuous map, multiple attractors may coexist. Also, in some cases at the bifurcation the fixed point is a saddle and no invariant circle is created.

There are four appendices. Appendix A contains proofs of lemmas and theorems that are deemed too lengthy for inclusion in the main text. The hope is that this segregation allows the general ideas of the book to be more easily assimilated. Extra figures intended as additional references are included in Appendix B. Appendix C overviews so-called adjugate matrices that are utilized throughout the book. Lastly Appendix D lists parameter values for the *S. cerevisiae* growth model studied in Chapter 4.

A quick note with regards to notation: $O(k)$ $[o(k)]$ will be used to denote terms that are order k or larger [larger than order k] in all variables and parameters of a given expression.

Finally, it should be noted that six research articles contain much of the work presented here. The codimension-one, discontinuous Hopf bifurcation discussed in Sec. 2.2 is the subject of [Simpson and Meiss (2007)]. Similarly, the coincidence of a Hopf bifurcation and a discontinuous bifurcation is described in [Simpson and Meiss (2008b)] and the coincidence of a period-doubling bifurcation and a border-collision bifurcation is described in [Simpson and Meiss (2009b)]. The bifurcation analysis of the *S. cerevisiae* model studied in Chapter 4 is presented in [Simpson *et al.* (2009)]. The N-dimensional unfoldings of Sec. 3.1 and Sec. 3.2 may also be found in this paper. Most of Chapter 6 and Chapter 7 may be found in [Simpson and Meiss (2009a)] and [Simpson and Meiss (2008a)] respectively.

D. J. W. Simpson

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Chapter 1

Fundamentals of Piecewise-Smooth, Continuous Systems

A system of ordinary differential equations (ODEs), $\dot{x} = F(x)$, or a map, $x' = F(x)$, is said to be *piecewise-smooth* if the phase space can be partitioned into countably many regions where F has a different smooth (i.e. C^k for some $k \in \mathbb{N}$) functional form. Nonsmoothness occurs on codimension-one region boundaries which are called *switching manifolds*. The last two decades have seen an explosion of interest in piecewise-smooth systems. Some recent textbooks in this area include [di Bernardo *et al.* (2008a); Leine and Nijmeijer (2004); Zhusubaliyev and Mosekilde (2003); Tse (2003); Banerjee and Verghese (2001); Brogliato (1999)].

It is useful to classify piecewise-smooth, ODE systems by the severity of the discontinuities at the switching manifolds:

(1) **Hybrid systems.** The system is a combination of differential equations and maps [Johansson (2003); Van der Schaft and Schumacher (2000)]. Such systems naturally arise when modeling hard impact phenomena [Wiercigroch and De Kraker (2000); Brogliato (1999); Popp (2000); Blazejczyk-Okolewska *et al.* (1999)]. This is because impacts result in the sudden velocity change of an object; when velocity is a system variable, a map is required to execute a jump in phase space.

(2) **Filippov systems.** The vector field, F, is discontinuous at switching manifolds. These systems are named after Aleksei Filippov who was perhaps the first to resolve ambiguities arising from *sliding* which occurs when the vector field on both sides of a switching manifold points either towards or away from the switching manifold [Filippov (1964, 1988)].

(3) **Piecewise-smooth, continuous systems.** F is continuous throughout the phase space but the Jacobian of F is discontinuous

at the switching manifolds.

(4) **Systems with higher order discontinuities.** Derivatives of F up to some degree n are continuous but of degree $n+1$ are not. Systems of this type are rarely encountered. As a general rule they exhibit the same bifurcations as smooth systems except non-degeneracy conditions dependent on high-order derivatives of F may not be computable.

Piecewise-smooth maps may be classified in a similar manner but this is not done here since many interesting piecewise-smooth maps contain a square-root or 3/2-type singularity. Other formulations exist for non-smooth systems such as *mixed logic dynamical systems* - a very general class of systems that use logic components [Ferrari-Trecate *et al.* (2003); Bemporad and Morari (1999)], *linear complementarity systems* - systems with inequality constraints [Heemels and Brogliato (2003); Heemels *et al.* (2000)], systems with differential inclusions [Deimling (1992)], and systems with delay [Sieber (2006); Sieber *et al.* (unpublished); Barton *et al.* (2006); Senthilkumar and Lakshmanan (2005)].

In piecewise-smooth systems the interaction of invariant sets with switching manifolds often produces bifurcations that are forbidden in smooth systems. These are collectively known as *discontinuity induced bifurcations*. Whereas period-doubling cascades are common in smooth systems, in piecewise-smooth systems periodic orbits may undergo what are called period-adding sequences or transition directly to chaos. Dynamical behavior local to a discontinuity induced bifurcation is often determined purely by linear terms of an appropriate piecewise series expansion. Consequently invariants commonly grow in size linearly with respect to a bifurcation parameter, such as the Hopf cycle in the discontinuous analogue of an Andronov-Hopf bifurcation, see Sec. 2.2.

Much is known about discontinuity induced bifurcations in systems of low dimension ($N = 1, 2$). However, the technique of dimension reduction via center manifold analysis that is hugely useful in studying smooth systems does not apply at non-differentiable points in piecewise-smooth systems. More dimensions provide more allowable geometries and only scattered results exist for such bifurcations in systems of any dimension. Furthermore, in piecewise-smooth, continuous, ODE systems, invariant manifolds of equilibria may be conical instead of planar [Carmona *et al.* (2005b)].

Perhaps the most basic discontinuity induced bifurcations are those that arise from the perturbation of a piecewise-smooth, ODE system that has an equilibrium located exactly on a switching manifold. Such bifurcations are commonly referred to as *boundary equilibrium bifurcations*. In hybrid or Filippov systems these bifurcations may involve what are known as *pseudo-equilibria* due to a lack of continuity in the vector field (these are not discussed in this book, the reader is referred to [di Bernardo *et al.* (2008a); Kuznetsov *et al.* (2003); Giannakopoulos and Pliete (2001); di Bernardo *et al.* (2008b); Buzzi *et al.* (2006)]). In piecewise-smooth, continuous systems, they are further known as *discontinuous bifurcations* and are either analogues of familiar smooth bifurcations, or novel and unique to piecewise-smooth systems. The discrete analogue, that is a bifurcation resulting from the perturbation of a piecewise-smooth, continuous map with a fixed point on a switching manifold, is known as a *border-collision bifurcation*.

The remainder of this chapter is organized as follows. Section 1.1 presents an overview of physical systems that have been modeled by piecewise-smooth systems. Section 1.2 introduces a general form commonly used to investigate discontinuous and border-collision bifurcations. Here it is shown that nonlinear terms do not affect structurally stable local dynamics. Equilibria and fixed points of the general form are then computed in Sec. 1.3. When particular non-degeneracy conditions are satisfied, the general form may be put into a canonical form involving companion matrices, Sec. 1.4. Section 1.5 summarizes current theory relating to discontinuous bifurcations. The discrete equivalent, border-collision bifurcations, are discussed in Sec. 1.6. Section 1.7 describes Poincaré maps of piecewise-smooth, ODE systems. Generically, the Poincaré maps are smooth unless a tangency occurs, in which case so-called discontinuity maps are used to control any singularities. Four common tangency scenarios are described in detail. The concept of sliding is also discussed in this section. Section 1.8 describes period-adding cascades and finally Sec. 1.9 discusses smooth approximations to piecewise-smooth systems.

1.1 Applications

This section discusses applications of piecewise-smooth systems. To be concordant with the focus of this book an emphasis is placed on physical systems that are well-modeled by piecewise-smooth, continuous, ODE systems and maps. However, also of interest are piecewise-smooth, discontinuous, ODE systems that exhibit oscillatory behavior and are of a form that leads to piecewise-smooth, continuous Poincaré maps. As detailed in