GOTTLOB FREGE

Collected Papers on Mathematics, Logic, and Philosophy

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EDITOR'S PREFACE

Gottlob Frege's occasional writings are all collected in the present volume, about half of them now first Englished. They form a corpus almost as great and as important as his three systematic works. The early mathematical writings, including the dissertations that won him his doctorate and his *venia docendi*, show what goals and methods appealed to him as a mathematician: they deserve study by anyone investigating the motivation and the starting-point of Frege's great works on the foundations of mathematics. The essays on the philosophy of language belonging to his middle period are yet more significant: they provide, it might be said, the foundation of the foundations. Here we have them in full, together with Frege's controversial writings and his attempt to come to terms with the views of Husserl, Cantor, and others. From a later period we have the two series of articles on the foundations of geometry, the topic to which he turned after his work on the foundations of arithmetic had reached a conclusion that would have seemed triumphant but for the perhaps fatal flaw on which Russell struck his finger. From the last years of Frege's life come the three chapters (all that were published) of his 'Logical Investigations', as fresh an approach to the philosophy of logic and of language as can be imagined. But it is impertinent to praise Frege: the purpose of the present remarks is only to point out that this volume contains some of his most characteristic works.

It reproduces in English and was made possible by I. Angelelli's valuable edition of *Kleine Schriften* (Olms, Hildesheim, 1967), omitting only material now available in *Philosophical and Mathematical Correspondence* (Blackwell, 1980) or *Posthumous Writings* (Blackwell, 1979). It and they, together with *The Foundations of Arithmetic* (latest edition Blackwell, 1980) and *Conceptual Notation and related articles* (Oxford University Press, 1972), constitute almost a complete English Frege. There wants only a translation of *Grundgesetze der Arithmetik*: Montgomery Furth's *The Basic Laws of Arithmetic* (California University Press, 1964) was only a partial translation and is now, alas, out of print.

Marginal numbers give the pagination of the original for each item and a marker in the text indicates the original page-division as accurately as possible.

Sincere thanks are due to the editors and translators, who have permitted their collections or individual publications to be subsumed into the present volume, to Max Black, Victor Dudman, Peter Geach, E.-H. W. Kluge, and R. H. Stoothoff. All have collaborated most helpfully in the gentle harmonization that was thought proper to this publication in collected form. The *Australasian Journal of Philosophy* and *Mind* kindly gave permission to publish Mr. Dudman's translations, which appeared in their 47th and 79th volumes in 1969 and 1970 respectively; Yale University Press those of Mr. Kluge, which appeared in Gottlob Frege *On the Foundations of Geometry and Formal Theories of Arithmetic* (1971).

As in most previous publications from this house '*Bedeutung*' has throughout been rendered by 'meaning' and '*Satz*' generally bý 'proposition', though the 'sentence' of earlier translators has been retained in some later writings. In the mathematical translations an attempt has been made both to use a contemporary terminology and to retain Frege's concreteness and his imagery, at the cost (not thought excessive) of some divergence from how the same content would be expressed by a modern mathematician. Here Hans Kaal and I have once again profited alike by the sympathy and by the breadth of knowledge of my colleague Peter Neumann. William Ewald, also a colleague, generously read the early papers for us with his logician's eye. Hans Kaal himself, the translator of nearly all the newly Englished material, I may be allowed to thank for a conscientiousness and a willingness to revise that would perhaps have seemed wasted on a lesser author.

B. McG.

On a Geometrical Representation of Imaginary Forms in the Plane

When we consider that the whole of geometry rests ultimately on axioms which derive their validity from the nature of our intuitive faculty, we seem well justified in questioning the sense of imaginary forms, since we attribute to them properties which not infrequently contradict all our intuitions.

By way of comparison let us take forms at infinity, which do not occur in the space of our intuition either. Taken literally, a 'point at infinity' is even a contradiction in terms; for the point itself would be the end point of a distance which had no end. The expression is therefore an improper one, and it designates the fact that parallel lines behave projectively like straight lines passing through the same point. 'Point at infinity' is therefore only another expression for what is common to all parallels, which is what we commonly call 'direction'. As a straight line is determined by two points, it is also | given by a point 4
and a direction. This is only an instance of the general law that, whenever we are dealing with projective relationships, a direction can represent a point. By designating the direction as a point at infinity, we forestall a difficulty which would otherwise arise because of the need to distinguish a frequently unsurveyable set of cases according to whether two or more of the straight lines in the set were parallel or not. But once the principle of the equivalence of direction and point is established, all these cases are disposed of at one blow.

The situation for imaginary forms is quite similar. If we calculate, e.g., the points of intersection of a circle with a straight line which does not meet the circle, we get conjugate complex expressions for the coordinates. The meaning of imaginary points so defined is not confined to the field of algebraic analysis. For a straight line is related to all circles which together with it determine the same imaginary points in the very same way as it is related to a system of circles which have the same real points in common with it. Circles are also related to one another as if they had two points in common; e.g., they form an involution on the straight line that joins their centres. Imaginary points

Doctoral dissertation in the Philosophical Faculty of Göttingen (A. Neuenhann, Jena, 1873). (*Tr.* Hans Kaal)

can therefore be defined also in purely geometrical terms by combination of a circle with a straight line or by involution on a straight line. Moreover, the property of a conic, that of being a circle, is equivalent
5 to the property of its passing through two points. E.g., while | a conic is in general determined by five points, three will suffice for a circle. Imaginary points therefore resemble points at infinity in that they too express something common to several forms. Thus the imaginary circular points at infinity are the common characteristic marks by which circles are distinguished from all other curves. Analytically this common element is represented in the form of complex coordinates which satisfy the equation of the curve. Since we can perform the same operations with these complex numbers as with real ones, we can infer a set of geometrical propositions from these imaginary points of intersection which we could also infer from the real ones.

It is now of the greatest importance to find out when a proposition which holds for real forms can be carried over to imaginary ones. In order to answer this question we must recall the foundations of analytic geometry. The equation of a straight line is derived with the help of propositions about the similarity of triangles and about the angles formed by parallel lines. From the same propositions we can infer Pythagoras's theorem. which in turn gives us an expression for the distance between two points. These are the elements from which all geometrical constructions are composed. Conversely, the propositions about similarity and about angles formed by parallel lines, and hence also all the propositions that follow from them, can also be proved by analytic geometry. If the operations and inferences performed in a
6 particular case are also applicable to complex numbers, | then the proposition can be extended to imaginary forms. On the other hand, a truth requiring additional operations or inferences for its proof does not in general hold for imaginary forms. Now with few exceptions all the operations and concepts that occur in the case of real numbers can indeed be carried over unchanged to complex ones. However, the concept of being greater cannot very well be applied to complex numbers. In the case of integration, too, there appear differences which rest on the multplicity of possible paths of integration when we are dealing with complex variables. Nevertheless, the large extent to which imaginary forms conform to the same laws as real ones justifies the introduction of imaginary forms into geometry.

As in the consideration of points at infinity, there now arises the need, not only for treating these improper elements in the same way as the proper ones, but also for having them before our eyes. This is easily achieved for points at infinity in the plane by projecting the plane on a

sphere from a point on the sphere which is neither the nearest nor the furthest. In that case there is no difference in projection between proper points and points at infinity. In what follows we shall attempt to do the same for imaginary forms. By a geometrical representation of imaginary forms in the plane we understand accordingly a kind of correlation in virtue of which every real or imaginary element of the plane has a real,
| intuitive element corresponding to it. The first advantage to be gained 7
by this is one common to all cases where there is a one–one relation between two domains of elements: that we can arrive at new truths by merely carrying over known propositions. But there is another advantage peculiar to this case: that the non-intuitive relations between imaginary forms are replaced by intuitive ones. The meaning of imaginary forms comes out equally whether they are considered metrically or projectively. We shall, however, confine ourselves to metrical relations and only indicate at the end a way of generalizing our method of representation which might be more suitable for projective propositions.

1 REPRESENTATION OF IMAGINARY POINTS

For the sake of brevity and precision of expression in what follows, we introduce the following designations:

The plane whose forms we represent shall be called the base plane. The points, straight lines and curves to be represented shall always be distinguished by the addendum 'real' or 'imaginary' from those forms that serve to represent them and which are always to be regarded as real. Further, the real shall in general be subsumed under the imaginary. Now if we want to represent imaginary points on the base plane, it seems appropriate that we start with the way they are defined in algebraic analysis, because this allows us to describe them in their entirety in the most general way. Accordingly, we think of | imaginary 8
points as given by their rectangular coordinates

$$x = \xi + i\xi', \qquad y = \eta + i\eta'.$$

We could now represent the imaginary points (x, y) by the two points (ξ, η) and (ξ', η'). However, this would not enable us to tell which one of the two was to express the real parts. We therefore displace the point (ξ', η') onto a special plane which is parallel to the base plane and contains the rectangular axes of ξ' and η'. We call this plane the plane of the imaginary. The base plane considered as the locus of the points (ξ, η) may be called the plane of the real. In order to characterize as such those points on these planes that belong together, we connect them by a straight line, and we regard this as a representation of the imaginary point. If a straight line passes through the origin of coordinates in the plane of the imaginary, it represents a real point. For the sake of brevity we will call the origin of these coordinates the point of origin of the imaginary.

2 IMAGINARY CURVES AND IN PARTICULAR THE IMAGINARY STRAIGHT LINE

Let some curve be given by the equation

$$S(x, y) = 0.$$

This breaks down into

$$\varphi(\xi, \xi', \eta, \eta') = 0, \qquad \psi(\xi, \xi', \eta, \eta') = 0. \tag{1}$$

Each system of values ξ, ξ', η, η' which satisfies equations (1) gives us
9 an imaginary point of the curve. | There is a doubly infinite set of such points, if we call the multiplicity of real points on a straight line a singly infinite set. If we solve equations (1), we obtain

$$\xi' = f(\xi, \eta), \qquad \eta' = f_1(\xi, \eta)$$

and these functions give us a mapping of the plane of the real to the plane of the imaginary.

Suppose first that the curve is a straight line given by the equation

$$ux + vy + 1 = 0,$$

where $$u = \rho + i\rho', \qquad v = \chi + i\chi'.$$

The mapping functions are then:

$$\begin{aligned} \xi' &= \frac{\chi + (\rho\chi + \rho'\chi')\xi + (\chi^2 + \chi'^2)\eta}{\rho'\chi - \chi'\rho} \\ \eta' &= \frac{-\rho - (\rho^2 + \rho'^2)\xi - (\rho\chi + \rho'\chi')\eta}{\rho'\chi - \chi'\rho} \end{aligned} \tag{2}$$

or in the case of the inverse solution:

$$\begin{aligned} \xi &= \frac{\chi' - (\rho\chi + \rho'\chi')\xi' + (\chi^2 + \chi'^2)\eta'}{\rho'\chi - \chi'\rho} \\ \eta &= \frac{-\rho' + (\rho^2 + \rho'^2)\xi' + (\rho\chi + \rho'\chi')\eta'}{\rho'\chi - \chi'\rho} \end{aligned} \tag{3}$$

The mapping is one–one.

Not every pair of linear functions

$$\xi' = A + B\xi + C\eta, \qquad \eta' = D + E\xi + F\eta$$

can occur in formulae (2). The following conditions must first be satisfied:

$$F + B = 0, \qquad BF - EC = 1. \tag{4}$$

In order to investigate what special kind of mapping may be given in this way, we effect a rotation of the coordinate system in both planes by putting: |

$$x = x_1 \cos\alpha - y_1 \sin\alpha, \qquad y = x_1 \sin\alpha + y_1 \cos\alpha.$$

We then obtain the new mapping functions

$$\xi'_1 = A_1 + B_1\xi_1 + C_1\eta_1, \qquad \eta'_1 = D_1 + E_1\xi_1 + F_1\eta_1.$$

Here

$$B_1 = B\cos^2\alpha + F\sin^2\alpha + (E + C)\sin\alpha\cos\alpha$$

or

$$B_1 = B\cos 2\alpha + \frac{E+C}{2}\sin 2\alpha$$

$$F_1 = -B\cos 2\alpha - \frac{E+C}{2}\sin 2\alpha.$$

We can now determine α in such a way that $B_1 = 0$ and $F_1 = 0$. We then have

$$\operatorname{tg} 2\alpha = -\frac{2B}{C+E}.$$

From this we can infer two values for α itself which differ from each other by 90°. We now have

$$\xi'_1 = A_1 + C_1\eta_1, \qquad \eta'_1 = D_1 + E_1\xi_1. \tag{5}$$

Equation (4) can be transformed into

$$E_1 C_1 = -1. \tag{6}$$

Now what is the geometrical meaning of this? To a parallel to the ξ_1-axis corresponds a parallel to the η'_1-axis, and to a parallel to the η-axis a parallel to the ξ'_1-axis. Up to now we have made no stipulations about the respective positions of the coordinate systems in the planes of the real and the imaginary. We now stipulate that the coordinate system in the plane of the imaginary has been rotated through 90°

relative to the one in the plane of the real, so that the ξ'-axis is parallel to the η-axis, and the η'-axis to the negative side of the ξ-axis. The
11 same holds then for ξ'_1 and η'_1 with respect | to ξ_1 and η_1. The advantage to be gained by this is that the parallels to the ξ_1- and η_1-axes become parallel to their images. It is now possible to carry out the mapping by means of a simple geometrical construction. For if we place a plane through each pair of parallels to the η_1- and ξ'_1-axes, then all these planes intersect in a common edge NR (figure 1), because in virtue of (5) the distance between two parallels in the plane of the real has a constant relation to the distance between their images. In the same way, all of the planes placed through each pair of parallels to the ξ_1- and η_1-axes intersect in a common edge QM. These edges are parallel to the base plane and perpendicular to each other. In virtue of equations (5) and (6), and because the axes of η'_1 and ξ_1 are opposite whereas those of η_1 and ξ'_1 have the same direction, the edge QM must be at the same distance from the plane of the real as the edge RN is from the plane of the imaginary, but in the opposite direction. The mapping is given by the edges RN and QM; for every straight line which connects a point on NR with a point on QM cuts the two planes at corresponding points. Since the two edges yield in this way all the

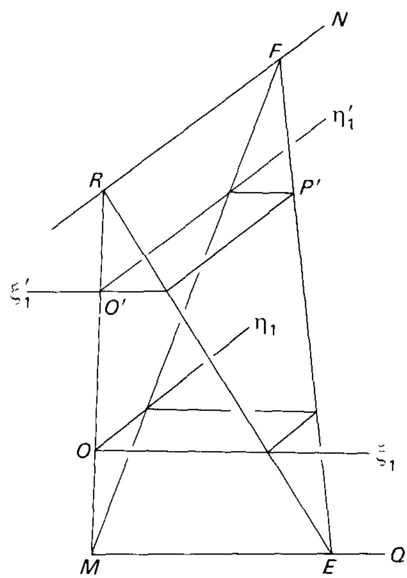

FIGURE 1

imaginary points on the imaginary straight line, we regard them as a representation of the latter. For a pair of straight lines to represent an imaginary straight line, they must be perpendicular to each other and parallel to the base plane, and one of them must be just as distant from the | plane of the real as the other is from the plane of the imaginary, 12
but in the opposite direction. We will call such a pair of straight lines the guide lines of an imaginary straight line. The special case of a real straight line still remains to be considered. An imaginary straight line has only one real point, for through the point of origin of the imaginary we can in general place only *one* straight line intersecting the two guide lines. Only when one of the guide lines itself passes through the point of origin of the imaginary do we have infinitely many real points. These are represented by the straight lines which connect them with the point of origin of the imaginary. Since one of the guide lines lies in the plane of the imaginary, the other must lie in the plane of the real, and consequently it must coincide with the real straight line to be represented. In addition to the real points, we have in this case a doubly infinite set of imaginary points, which are represented by the lines connecting other points on the guide line in the plane of the imaginary with points on the guide line in the plane of the real. We cannot properly speak of a mapping in this case, as is clear from the disappearance of the denominator in (2) and (3) and is also geometrically self-evident. This happens whenever the guide lines lie in the planes of the real and the imaginary, or in analytical terms, when the denominator $\rho'\chi-\chi'\rho$ of formulae (2) disappears. The two conditions are clearly identical; for if there is no possibility of a mapping, each of them | is necessary and sufficient. When $\rho'\chi=\chi'\rho$, we can put 13

$$u=\rho+i\rho'=Q(\cos\gamma+i\sin\gamma)$$

$$v=\chi+i\chi'=R(\cos\gamma+i\sin\gamma).$$

The coefficients u, v of the equation $ux+vy+1=0$ have therefore the same amplitude. We will call such an imaginary straight line a purely imaginary one. Considered metrically, such straight lines are closer to real lines.

3 THE IMAGINARY CONNECTING LINE

Suppose we are given two straight lines (g, h) which represent imaginary points, and we are to construct the guide lines of their imaginary connecting line.

We first exclude the case where g or h is parallel to the base plane and assume that the two intersect. Then the guide lines we are seeking must either lie in the plane of g and h or pass through their point of intersection. Both guide lines cannot lie in the plane of g and h, because they would then be parallel to each other. Hence one at any rate passes through the point of intersection. But then the other cannot in general pass through that point as well, because of the condition that must be satisfied by the distances of the guide lines from the planes. The second guide line lies therefore in the plane of g and h. This and the other restrictions to which it is subject determine its position completely. This
14 also gives us the direction | of the guide line which passes through the point of intersection of g and h. The only cases where neither of the guide lines needs to lie in the plane of g and h are those where the point of intersection of g and h is equidistant from the planes of the real and the imaginary, and hence, where it lies either in the middle between the two or at infinity. In these cases, both guide lines pass through the point of intersection and their direction becomes indeterminate. In the second place, if the straight line g is parallel to the base plane, it evidently represents an imaginary point at infinity. We do not assume that g is cut by h, but that g lies neither in the middle between the planes of the real and the imaginary nor in the plane at infinity. If we now place through g a plane F parallel to the base plane and draw in it a guide line which intersects g and h, then the other guide line cannot also lie in this plane. If it is nevertheless to intersect g and at the same time to be parallel to the base plane, it must be assumed to be parallel to g. This and the other conditions determine it completely. From this it follows that the first guide line intersects g at a right angle; it too is completely determined by this. g represents the imaginary point at infinity on the straight line whose guide lines we have just drawn. If g is displaced on the plane F parallel to itself, the construction remains unchanged. All those parallels represent accordingly the same imaginary point at infinity. If we place a plane F' parallel to the
15 base plane through the other guide line and draw in | it lines perpendicular to it, or what comes to the same thing, perpendicular to g, then these perpendiculars together with h yield the same imaginary point at infinity. Such a point is therefore represented by two singly infinite families of parallels which are perpendicular to one another and which lie in two planes which are parallel and symmetrical to the planes of the real and the imaginary. The case where g lies either parallel to and in the middle between the plane of the real and the imaginary or in the plane at infinity will be considered later on.

If h is also parallel to the base plane, then the imaginary line connecting the imaginary points at infinity represented by h and g is the line at infinity of the base plane. This is therefore one of the guide lines. The question about the other one must remain unanswered for the time being. We here exclude the cases where, according to the results obtained above, g and h represent the same point at infinity.

Finally, if g and h are neither parallel to the base plane nor intersect each other, then we project everything on a plane which lies in the middle between the planes of the real and the imaginary and parallel to them, and which will subsequently always be designated by E. Let the lines of projection be parallel to a straight line g. This is then mapped as the point G (figure 2). Let the point of intersection of the straight line h with the plane E be H and its projection h_1. Further, let GA
and GB be the projections of the guide lines we are seeking. | Then 16
$HA = HB = Hg$. We thus obtain A and B by marking off HG from H to h in both directions. The projections GA and GB we have found can then be moved to their positions in space by parallel displacement along g.

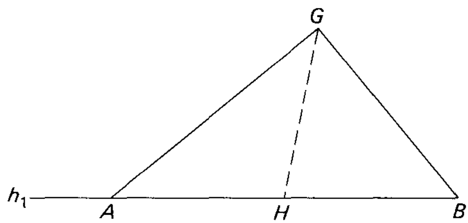

FIGURE 2

4 THE DISTANCE BETWEEN IMAGINARY POINTS

The expression

$$r = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \tag{1}$$

expresses the distance between two points when x_0, y_0 and x_1, y_1 are the real coordinates of these points. Every proposition which states a relation between lengths and does not merely contain an inequality follows from the foundations of analytic geometry and can be derived analytically from (1) by operations and inferences which are equally applicable to complex numbers. These relations obtain therefore also among the values of r for complex coordinates. If we now take the view that what is essential to the concept of distance is not the intuitive character of a straight line but conformity to the laws of algebraic analysis, then we can apply the name 'distance' also where the end points are imaginary. We shall henceforth speak of 'distances' in this sense.

If we now introduce into (1) the values

$$x_0=\xi_0+i\xi'_0, \qquad y_0=\eta_0+i\eta'_0$$
$$x_1=\xi_1+i\xi'_1, \qquad y_1=\eta_1+i\eta'_1,$$

we obtain |

17
$$r=\sqrt{\begin{matrix}(\xi_1-\xi_0)^2+(\eta_1-\eta_0)^2-(\xi'_1-\xi'_0)^2-(\eta'_1-\eta'_0)^2\\ +2i[(\xi_1+\xi_0)(\xi'_1-\xi'_0)+(\eta_1-\eta_0)(\eta'_1-\eta'_0)]\end{matrix}}$$

or, by introducing

$$\xi_1-\xi_0=\sigma, \qquad \eta_1-\eta_0=\tau$$
$$\xi'_1-\xi'_0=\sigma', \qquad \eta'_1-\eta'_0=\tau',$$
$$r=\sqrt{\sigma^2+\tau^2-\sigma'^2-\tau'^2+2i(\sigma\sigma'+\tau\tau')}. \tag{2}$$

To simplify the calculation, we place the coordinate system in such a way that its axes becomes parallel to the guide lines of the imaginary connecting line between (x_0, y_0) and (x_1, y_1). As in section 3, we again project everything on the plane E in the way indicated in that section. The straight line g, which represents the imaginary point (x_0, y_0), is mapped as the point G (figure 3). Let h_1 be the projection of h, the straight line that represents the imaginary point (x_1, y_1). GB and GA are the projections of the guide lines. C and D are the images of the intersections of h with the planes of the real and the imaginary. Then $CA=DB$. If we now draw $CK\|DJ\|AG$, then

$$\begin{aligned} GJ&=\xi_1-\xi_0=\sigma\\ DJ&=\eta_1-\eta_0=\tau\\ GK&=-(\eta'_1-\eta'_0)=-\tau'\\ CK&=\xi'_1-\xi'_0=\sigma'. \end{aligned} \tag{3}$$

Further, $\dfrac{DJ}{JB}=\dfrac{CK}{BK}$ or $\dfrac{DJ}{GK}=\dfrac{CK}{GJ}$ or

$$\frac{\tau}{-\tau'}=\frac{\sigma'}{\sigma}. \tag{4}$$

18 If for σ in (2) we introduce the value $-(\tau'/\tau)\sigma'$, | we obtain for the real

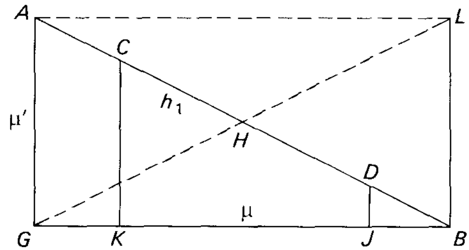

FIGURE 3

part of the expression under the root sign

$$A = \frac{(\sigma'^2 - \tau^2)(\tau'^2 - \tau^2)}{\tau^2}$$

and as the factor of i

$$B = -\frac{2\tau'}{\tau}(\sigma'^2 - \tau^2).$$

Now if

$$r = \rho + i\rho' = \sqrt{A + iB},$$

then

$$\rho = \sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}}, \qquad \rho' = \sqrt{\frac{\sqrt{A^2 + B^2} - A}{2}}.$$

Here $\sqrt{A^2 + B^2}$ is always to be taken as positive, and ρ and ρ' have the same sign if $B > 0$ and the opposite sign if $B < 0$. By substituting the values of A and B we get

$$\rho = \sqrt{\tau^2 - \sigma'^2}, \qquad \rho' = \frac{\tau'}{\tau}\sqrt{\tau^2 - \sigma'^2},$$

if $\sigma'^2 < \tau^2$, or

$$\rho = \frac{\tau'}{\tau}\sqrt{\sigma'^2 - \tau^2}, \qquad \rho' = -\sqrt{\sigma'^2 - \tau^2},$$

if $\sigma'^2 > \tau^2$.

The case where $\tau^2 > \sigma'^2$ can always be reduced to the case where $\sigma'^2 > \tau^2$ by rotation of the coordinate system through 90°. Such a rotation does not disturb the assumed parallelism between the axes and the guide lines. If we distinguish the new σ and τ by a subscript, then |

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$$\tau=\sigma_1$$
$$\sigma'=-\tau'_1$$
$$\sigma=-\tau_1$$
$$\tau'=\sigma'_1.$$

Thus if

$$\tau^2>\sigma'^2 \quad \text{or}$$
$$\tau^2\sigma^2>\sigma^2\sigma'^2$$

or

$$\tau'^2\sigma'^2>\sigma^2\sigma'^2$$
$$\tau'^2>\sigma^2,$$

then

$$\sigma_1'^2>\tau_1{}^2.$$

We can therefore always presuppose the case where $\sigma'^2>\tau^2$. Then

$$r=\sqrt{\sigma'^2-\tau^2}\left(\frac{\tau'}{\tau}-i\right).$$

This leads us to compare the real and imaginary parts of this formula with the segments $GB=\mu$, $GA=\mu'$ on the guide lines. Then (figure 3)

$$\mu=GJ+GK, \qquad \mu'=KC+DJ$$

or by (3)

$$\mu=\sigma-\tau', \qquad \mu'=\sigma'+\tau$$

and in the light of (4)

$$\mu=-\frac{\tau'}{\tau}(\sigma'+\tau).$$

Accordingly

$$\rho=-\mu\sqrt{\frac{\sigma'-\tau}{\sigma'+\tau}}$$

$$\rho'=-\mu'\sqrt{\frac{\sigma'-\tau}{\sigma'+\tau}}.$$

Consequently

$$r = \pm \sqrt{\frac{\sigma' - \tau}{\sigma' + \tau}} (\mu + i\mu'), \quad | \tag{5}$$

where the sign of the root depends on arbitrary assumptions. Given our 20
assumption that $\sigma'^2 > \tau^2$, the root is always real. It assumes a simple form when the connecting line is a purely imaginary straight line. Then the guide lines lie in the planes of the real and the imaginary. If we assume the ξ-axis to be parallel to the guide line in the plane of the real, then the condition that $\sigma'^2 > \tau^2$ is satisfied, since $\tau = 0$. Formula (5) then becomes

$$r = \mu + i\mu'.$$

This leads one to conjecture that the factor

$$\sqrt{\frac{\sigma' - \tau}{\sigma' + \tau}}$$

depends on the distance of the guide lines from the planes of the real and the imaginary. For the sake of symmetry we determine the distance N of the guide lines from the plane E and call $2d$ the distance of the planes of the real and the imaginary from each other. Then (figure 4)

$$\frac{N + d}{N - d} = \frac{CD}{AB} = \frac{\xi'_1 - \xi_0}{\eta_1 - \eta_0} = \frac{\sigma'}{\tau},$$

where the plane of the drawing has been placed through the η- and ξ'-axis. From this we can infer the following relationship between the

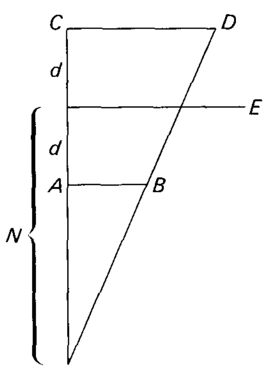

FIGURE 4

distances:

$$\lambda = \frac{N}{d} = \frac{\sigma' + \tau}{\sigma' - \tau}.$$

Formula (5) becomes

$$r = \frac{1}{\sqrt{\lambda}}(\mu + i\mu').$$

If we consider once more the projection on the plane *E* (figure 3), we notice that, according to Gauss's way of representing complex numbers in the plane, the diagonal *GL* of the rectangle constructed from the sides

$$GB = \mu, \qquad GA = \mu' \quad |$$

21 gives us the distance we are seeking up to a real factor, if we assume the axis of the real to be parallel to the one guide line and the axis of the plane of the imaginary to be parallel to the other guide line. The same holds also for *GH*, the distance between the points of intersection of the straight lines *g* and *h* with the plane *E*, because $GH = \frac{1}{2}GL$. It still remains to be investigated to which of the two guide lines the axis of the real is to be made parallel. It is to be made parallel to that guide line to which we had to make the ξ-axis parallel, in order to satisfy the condition

$$\sigma'^2 > \tau^2;$$

for this assumption lies at the bottom of all our considerations. Now according to (3) this inequality is identical with $CK^2 > DJ^2$. In view of figure 3 we can substitute for it

$$DB^2 > CB^2. \tag{6}$$

Now *DB* and *CB* are related in the same way as the distances between the guide line which is parallel to the ξ-axis and the planes of the real and the imaginary. Accordingly the inequality (6) expresses that we must assume the ξ-axis to be parallel to that guide line which is nearer in the absolute sense to the plane of the real. Accordingly we obtain the proposition:

> If the imaginary points γ and δ on the imaginary straight line ε are represented by the straight lines *g* and *h*, then the connecting line *GH* of the points of intersection of *g* and *h* with the plane *E*, multiplied by a real constant $2/\sqrt{\lambda}$ dependent only
> 22 on ε, | gives us an intuitive representation of the imaginary