AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XXIV

STRUCTURE OF ALGEBRAS

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PREFACE

The theory of linear associative algebras probably reached its zenith when the solution was found for the problem of determining all rational division algebras. Since that time it has been my hope that I might develop a reasonably self-contained exposition of that solution as well as of the theory of algebras upon which it depends and which contains the major portion of my own discoveries. The first step in carrying out this desire was necessarily that of writing a text with contents selected so as to provide a foundation adequate for the prospective exposition. This text has already been published under the title *Modern Higher Algebra*. Its completion was followed shortly by the timely invitation of the Colloquium Committee of the American Mathematical Society to me to write these LECTURES embodying the desired exposition.

It has been most fortunately possible at this time to give a new treatment of the early parts of our subject simplifying not only the proofs in the theory of normal simple algebras but even the exposition of the structure theorems of Wedderburn. This does not evidence itself in the somewhat classical first two chapters. These contain the preliminary discussion of linear sets, direct products, direct sums, ideals in an algebra, and similar topics, with the additions and modifications made necessary by the fact that we are considering here algebras over an arbitrary field and that inseparable extension fields may exist. But the exposition given of the usual fundamental theorem, stating that every linear associative algebra is equivalent to a first algebra of square matrices and reciprocal to a second such algebra, is expanded here so as to have as consequence a result basic in the new treatment of the Wedderburn structure theory. While this result is not derived until its need appears in Chapter III the proof is so elementary that it might have been placed in the first chapter without change.

This basic theorem is that of R. Brauer on the structure of the direct product of a normal division algebra and its reciprocal algebra. It is combined with two theorems of J. H. M. Wedderburn to obtain as generalizations three tool theorems which are used throughout Chapters III and IV, and yield rather remarkable simplifications of the proofs of numerous fundamental results. In particular in Chapter III the foundation of the proof of the Wedderburn principal theorem on the structure of an algebra with a radical is simplified and the theorem itself then obtained.

While the first three chapters of this exposition do contain a considerable amount of new material their principal content is evidently an exposition in more modern form of the Wedderburn structure theorems which were first presented in book form in the two editions of the text on our subject by L. E. Dickson. I owe much to these expositions as well as to their author, who has been my teacher and the inspiration of all my research. The remaining eight chapters of the present text are composed principally of results derived since 1926 when the second (German) edition of Dickson's text was written. These results are due principally to R. Brauer, H. Hasse, E. Noether, and myself. Their exposition is begun in Chapter IV which contains the theory of the commutator subalgebra of a simple subalgebra of a normal simple algebra, the study of automorphisms of a simple algebra, splitting fields, and the index reduction factor theory.

The fifth chapter contains the foundation of the theory of crossed products and of their special case, cyclic algebras. The theory of exponents is derived there as well as the consequent factorization of normal division algebras into direct factors of prime-power degree.

Chapter VI consists of the recent study of the abelian group of cyclic systems which is applied in Chapter VII to yield the theory of the structure of direct products of cyclic algebras and the consequent properties of norms in cyclic fields. This chapter is closed with the recently developed theory of p-algebras.

In Chapter VIII an exposition is given of the theory of the representations of algebras. The treatment is somewhat novel in that while the recent expositions have used representation theorems to obtain a number of results on algebras, here the theorems on algebras are themselves used in the derivation of results on representations. The presentation has its inspiration in my work on the theory of Riemann matrices and is concluded by an introduction to the generalization (by H. Weyl and myself) of that theory.

In the ninth chapter the structure of rational division algebras is determined. The study begins with a detailed exposition of Hasse's theory of p-adic division algebras. The results are then extended so as to yield the theorems on rational division algebras without recourse to the theory of ideals in the integral sets of such algebras. This is believed to be the first time the extension has been made in a really simple fashion. The method is a greatly desirable one as the previous treatments used the results of a very voluminous theory which we are able to omit. It is necessary, of course, to assume without proof certain existence theorems from the theory of algebraic numbers. These presupposed theorems are indicated precisely, and I hope to be able to include their proofs in a future text on the theory of algebraic numbers and the arithmetic of algebras.

The theory of involutorial simple algebras arose in connection with the study of Riemann matrices but is now a separate branch of the theory of simple algebras with structure theorems on approximately the same level as those on arbitrary simple algebras. This theory is derived in Chapter X both for algebras over general fields and over the rational field. The results are also applied in the determination of the structure of the multiplication algebras of all generalized Riemann matrices, a result which is seen in Chapter XI to imply a complete solution of the principal problem on Riemann matrices.

This final reference is but one item in the last chapter which contains an exposition of a number of special results. In particular there are given new derivations of the structure of all normal division algebras of degrees three and four over any field. References to sources for the whole text are given in this chapter as well as indications of the literature on the subject and an extensive bibliography.

It is my hope that the form of this exposition will make it useful as a text on the theory of linear associative algebras as well as for its obvious purpose as a source book for young algebraists. Much of any success that there may have been in keeping the exposition completely correct and clear is due to the work of Dr. Sam Perlis who read the manuscript critically in each of the stages of its preparation. He not only assisted in keeping the exposition free of error but frequently indicated improvements resulting in greater clarity. I give him my great thanks.

I appreciate also the kind assistance of Professor Nathan Jacobson who suggested the proofs of two of the theorems as well as that of Mr. Morris Bloom who assisted in the preparation of the bibliography and give thanks to Professor Saunders MacLane and Dr. Otto F. G. Schilling who were a critical audience for oral expositions of some of the proofs. Final thanks are due to Professor G. A. Bliss without whose encouragement the completion of these LECTURES would have been greatly delayed.

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March 5, 1939

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CHAPTER I

FUNDAMENTAL CONCEPTS

1. The notations. As was stated in the preface, our exposition is founded upon the author's *Modern Higher Algebra* and will use the notations, definitions, and theorems of that text. We shall also use script letters to represent sets of elements as well as the Gothic letters used in the former text.

The reference notation used in the *Modern Higher Algebra* will also be used here. Thus Theorem 7.5 will refer to Theorem 5 of Chapter VII, equation (4.29) to (29) of Chapter IV. A large number of references will be made to results of the *Modern Higher Algebra* by the use of the following notational device. We shall refer to Chapter AX, to Section A4.9, to Theorem A6.5, to equation (A5.28), or to page A196, and shall mean the respective references of our foundation text. References will also be made by title number to articles listed in the bibliography at the end of these LECTURES.

2. Linear sets over $\mathfrak{F}$. The theory of algebras is a theory of certain types of linear sets of finite order over a field $\mathfrak{F}$. The field $\mathfrak{F}$ will occupy the rôle of fundamental underlying coefficient field in our exposition, and we shall use the symbol $\mathfrak{F}$ with this as its meaning and without further restatement of the fact throughout these LECTURES. Later special types of fields $\mathfrak{F}$ will be considered, but, until we state otherwise, $\mathfrak{F}$ will be arbitrary.

The concept of a linear set $\mathfrak{A}$ of order n over $\mathfrak{F}$ is the usual one of Section A2.11. Thus in particular every linear set of order n over $\mathfrak{F}$ is equivalent to the set of all sequences $(\alpha_1, \cdots, \alpha_n)$ with α_i in $\mathfrak{F}$ and such that

$$(\alpha_1, \cdots, \alpha_n) + (\beta_1, \cdots, \beta_n) = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n),$$

$$\lambda(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)\lambda = (\lambda\alpha_1, \cdots, \lambda\alpha_n),$$

for all α_i, β_i and λ of $\mathfrak{F}$. We shall use the notations and properties of linear sets given in Section A2.11 and shall now obtain some additional results.

The linear set $\mathfrak{L}_0$ consisting of zero alone will be called the *zero set.* We shall write $\mathfrak{L}_0 = 0$ and say that $\mathfrak{L}_0$ has *order zero.* In what follows let $\mathfrak{A}$ be any linear set of order $n > 0$ over $\mathfrak{F}$. The *sum* $(\mathfrak{A}_1, \cdots, \mathfrak{A}_t)$ of linear subsets $\mathfrak{A}_i$ of $\mathfrak{A}$ is defined as the set of all $a_1 + \cdots + a_t$ for a_i in $\mathfrak{A}_i$. It is easily verified to be a linear subset over $\mathfrak{F}$ of $\mathfrak{A}$. Addition of linear sets is then an associative and commutative operation.

The intersection $[\mathfrak{B}, \mathfrak{C}]$ of two linear subsets of $\mathfrak{A}$ is the set of all quantities common to $\mathfrak{B}$ and $\mathfrak{C}$. It is clearly a linear subset over $\mathfrak{F}$ of $\mathfrak{A}$. Moreover, as we shall show, *the order of* $(\mathfrak{B}, \mathfrak{C})$ *is the sum of the orders of* $\mathfrak{B}$ *and* $\mathfrak{C}$ *minus the order of* $[\mathfrak{B}, \mathfrak{C}]$.

The expression of the quantities of $\mathfrak{A} = (\mathfrak{A}_1, \cdots, \mathfrak{A}_t)$ in the form $a =$

$a_1 + \cdots + a_t$ with a_i in $\mathfrak{A}_i$ is unique if and only if the order of $\mathfrak{A}$ is the sum of the orders of the $\mathfrak{A}_i$. This property is true if and only if $a = 0$ implies that every $a_i = 0$. In this case we call the $\mathfrak{A}_i$ *supplementary* in their sum $\mathfrak{A}$, call $\mathfrak{A}$ the ***supplementary sum*** of the $\mathfrak{A}_i$, and write

$$\mathfrak{A} = \mathfrak{A}_1 + \cdots + \mathfrak{A}_t. \tag{1}$$

We shall not use the symbol $+$ for the sum of linear sets except when it is a supplementary sum. Observe that (1) holds if and only if $[\mathfrak{B}_i, \mathfrak{A}_{i+1}] = 0$ $(i = 1, \cdots, t)$, where $\mathfrak{B}_i = (\mathfrak{A}_1, \cdots, \mathfrak{A}_i)$.

In the exercise of Section A2.11 a linear set $\mathfrak{A} = (u_1, \cdots, u_n)$ over $\mathfrak{F}$ was considered with a proper linear subset $\mathfrak{B} = (v_1, \cdots, v_m)$ and the statement was made that then $\mathfrak{A} = (v_1, \cdots, v_n)$ over $\mathfrak{F}$ where $v_{m+1}, \cdots, v_n$ are in $\mathfrak{A}$. This result is important though trivially proved, and we state it as follows: *Let $\mathfrak{B}$ be a linear subset of $\mathfrak{A}$ over $\mathfrak{F}$. Then $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ where $\mathfrak{C}$ is a linear subset of $\mathfrak{A}$ called a supplement of $\mathfrak{B}$ in $\mathfrak{A}$.* Note that $\mathfrak{C}$ is not unique and that in particular $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}'$ where

$$\mathfrak{C}' = (v'_{m+1}, \cdots, v'_n), \qquad v'_i = v_i + b_i, \tag{2}$$

for any b_i in $\mathfrak{B}$.

To prove that the order of $(\mathfrak{B}, \mathfrak{C})$ is the sum of the orders of $\mathfrak{B}$ and $\mathfrak{C}$ minus the order of their intersection $\mathfrak{D}$, we have $\mathfrak{C} = \mathfrak{D} + \mathfrak{C}'$ as above. Then $(\mathfrak{B}, \mathfrak{C}) = (\mathfrak{B}, \mathfrak{C}')$. No quantity of $\mathfrak{C}'$ is in $\mathfrak{D}$ and hence no quantity of $\mathfrak{C}'$ is in $\mathfrak{B}$. It follows that $(\mathfrak{B}, \mathfrak{C}') = \mathfrak{B} + \mathfrak{C}'$, and the result we desire is an immediate consequence.

A set of quantities $b_1, \cdots, b_r$ of $\mathfrak{A}$ is said to *span* the linear subset $\mathfrak{B}$ of $\mathfrak{A}$ consisting of all $\lambda_1 b_1 + \cdots + \lambda_r b_r$ for λ_i in $\mathfrak{F}$. The order of $\mathfrak{B}$ is then an integer $m \leqq r$, and in fact the b_i may be renumbered so that $\mathfrak{B} = (b_1, \cdots, b_m)$ over $\mathfrak{F}$. Furthermore let $\mathfrak{A} = (u_1, \cdots, u_n)$ over $\mathfrak{F}$ and thus

$$b_i = \sum_{j=1}^{n} \lambda_{ij} u_j \qquad (i = 1, \cdots, r), \tag{3}$$

with unique λ_{ij} in $\mathfrak{F}$. Then m is the number of linearly independent b_i. By Exercise 5, page A67, m is the rank of the matrix $L = (\lambda_{ij})$. Also, if $r = n$, then $\mathfrak{B} = \mathfrak{A}$ if and only if $m = n$, that is, the square matrix L is non-singular.

The above is our final preliminary study and we shall now begin our treatment of algebras themselves.

3. Algebras over $\mathfrak{F}$. The theory of algebras of order n over $\mathfrak{F}$ was treated in Chapter AX from its aspect as a theory of matrices. While certain of the resulting properties are of considerable importance and will be used later we shall not base our present exposition upon that treatment but shall tend rather to a more abstract discussion.*

Algebras of finite order over $\mathfrak{F}$ may be defined as rings which are linear sets

* However, if the present treatment be used as a course text, the material of sections A10.1–5 is to be considered as *essential* introductory material.