STUDIES IN LOGIC

AND

THE FOUNDATIONS OF MATHEMATICS

VOLUME 83

NORTH-HOLLAND PUBLISHING COMPANY–AMSTERDAM · OXFORD
AMERICAN ELSEVIER PUBLISHING COMPANY, INC.–NEW YORK

LARGE INFINITARY LANGUAGES

MODEL THEORY

M. A. DICKMANN

C.N.R.S.
Université de Paris VII
Paris, France

1975

NORTH-HOLLAND PUBLISHING COMPANY–AMSTERDAM · OXFORD
AMERICAN ELSEVIER PUBLISHING COMPANY, INC.–NEW YORK

Library of Congress Catalog Card Number 73-81530
North-Holland ISBN S 0 7204 2200 0
0 7204 2281 7
American Elsevier ISBN 0 444 10622 7

Published by:
North-Holland Publishing Company–Amsterdam
North-Holland Publishing Company, Ltd.–Oxford

Sole distributors for the U.S.A. and Canada:
American Elsevier Publishing Company, Inc.
52 Vanderbilt Avenue
New York, N.Y. 10017

A mis hermanos latinoamericanos, victimas de la dictadura del capital, que combaten por una nueva sociedad

PREFACE

The present book grew out of my lecture notes "Model Theory of Infinitary Languages" published in January 1970 by the Mathematics Institute of Aarhus University, Denmark. Indeed, it is a thoroughly modified and updated version of those notes, which in its present form has only a vague resemblance to the original.

This book presents a systematic and largely self-contained development of the model theory of the infinitary languages $L_{\kappa\lambda}$ and $L_{\infty\lambda}$ (where κ,λ are infinite cardinals and $\infty \geqslant \kappa \geqslant \lambda$). The language $L_{\kappa\lambda}$ admits conjunctions and disjunctions of sets of formulas of power less than κ and simultaneous quantifications over sets of variables of power less than λ. $L_{\infty\lambda}$ is, of course, the "union" of the $L_{\kappa\lambda}$. The basic semantic notions are defined in Chapter 1 as direct generalizations of those for finitary, first order languages. The special case of $L_{\omega_1\omega}$ is not treated here; the word "large" in the title is intended precisely to denote this omission. The model theory of this particular language is the subject matter of Keisler's monograph [3].

The contents of the book can be briefly described as follows. Chapters 0 and 1 contain, for the most part, the necessary set-theoretic prerequisites and a list of the most basic notions, definitions, examples and notations used in the rest of the book; §2 is intended to give the reader something of a grasp on the "size" of large cardinals. I should mention that §5—"Partition Calculus"—was originally written jointly with George Rousseau; subsequently I made slight changes.

Chapter 2 contains a brief summary of the results from the model theory of $L_{\omega_1\omega}$ used in this book (§3), a more detailed presentation of some special topics from the model theory of $L_{\omega\omega}$ (§§1,2), and a list of examples showing the impossibility of extending certain results from $L_{\omega\omega}$ and $L_{\omega_1\omega}$ to other infinitary languages.

Thus, a considerable part of the first three chapters is devoted to the presentation of prerequisites and preparatory material.

Chapter 3 is concerned with the compactness problem (§§1,2) and a related analysis of various classes of large cardinals (§3), and with Löwenheim–Skolem type theorems (§§4,5). It contains several applications of the downward Löwenheim–Skolem theorem and the definition of Hanf and Morley numbers.

The central topic of Chapter 4 is the exact evaluation of the Hanf number of infinitary languages with finite quantification. The main result

—the Barwise–Kunen–Morley theorem—is approached through a sequence of steps which include, in. al., a detailed development of the omitting types technique, and the non-definability of well-orderings in the languages $L_{\kappa\omega}$.

Chapter 5 is devoted to languages with infinite quantification. In §1 we compute upper and lower bounds for the Hanf numbers of these languages in terms of large cardinals with partition properties. In §2 we study the preservation of infinitary equivalence under sums and products. §§3 and 4 are devoted to a comprehensive exposition of the method of extension of partial isomorphisms (the *back and forth method*), and to some of its applications. This method is an important research tool, both in logic and in algebra. The presentation here—"algebraic" rather than "game-theoretic"—shows that the infinitary languages $L_{\infty\lambda}$ are a natural logical framework for the standard forms of the back and forth method.

The book contains five appendices. Appendices A, D and E deal with particular topics from set theory which are either repeatedly used in the text or bear a close connection to parts of it. Appendices B and C deal with some constructions and results that require the coding of infinitary formulas; namely, the construction of a formula from $L_{\omega_1\omega_1}$ with no prenex form, and a survey of axiomatizability, definability and completeness results for infinitary languages, including a proof of Scott's undefinability theorem.

Special attention is devoted to the applications of the methods developed in the text to other topics in logic and mathematics. I believe that methods and ideas from model theory are powerful tools for purely mathematical research and that herein lies one of the most stimulating aspects of the subject; some results obtained in the 1960's serve to support this belief.

This monograph is conceived both as a reference book for researchers and as a textbook for graduate students. It can be used, partially or totally, in one- or two-semester graduate courses. In this case the student should be familiar with some basic results and methods from set theory and from the model theory of first order languages; familiarity with, roughly speaking, Part I of Keisler's book [3] is also recommended. §1 of Chapter 1 and §3 of Chapters 1 and 2, with their exercises, give a concise idea of the results with which familiarity is required.

Some indications concerning the way of reading this book: theorems, lemmas, examples, etc., are enumerated in consecutive order by three figures indicating, respectively, the chapter, the section, and the number of the item. Remarks and observations are numbered only when repeatedly mentioned elsewhere. Exercises are provided at the end of most sections, and have a separate numbering also by three figures. Previous results used in a proof are always explicitly mentioned. The end of a proof is indicated by the symbol ■. The logical interdependencies among the various parts of the book are illustrated in the chart following the table of contents.

I wish to express my deepest gratitude to several friends, students, colleagues and institutions whose generous cooperation and help made possible this work and who, in various ways contributed to improve the quality of its contents and style. In the first place to the Aarhus Mathematics Institute which on several occassions exempted me from other duties to allow me to write this book, and which generously contributed its printing facilities and the time of its typing staff. Within this institution I am specially indebted with Torben Larsen, for his unfailing help with writing and producing my earlier lecture notes on the subject of infinitary languages; with Brian Mayoh, who encouraged me to write those notes and saw to their publication; and with Lissi Daber and Ursula Engelke for their dedication and effort in producing the typescript both of the lecture notes and of large parts of the present book. Among my other friends and colleagues I shall specially acknowledge the generous help of John Bell, who read the entire manuscript and suggested numerous corrections, improvements of style and additions, and of Richard Gostanian, who read parts of the manuscript and also corrected points of style and suggested additions; the help of Kenneth McAloon was invaluable at the proof-correction stage. My warmest thanks are due to several colleagues, mentioned in the text, who made available unpublished results and permitted their inclusion in the book. Finally, my thanks to the editorial board of the series "Studies in Logic and the Foundations of Mathematics," and to Einar Fredriksson and Thom van den Heuvel of the staff of this series, for their patience, effort and help throughout the production of this book.

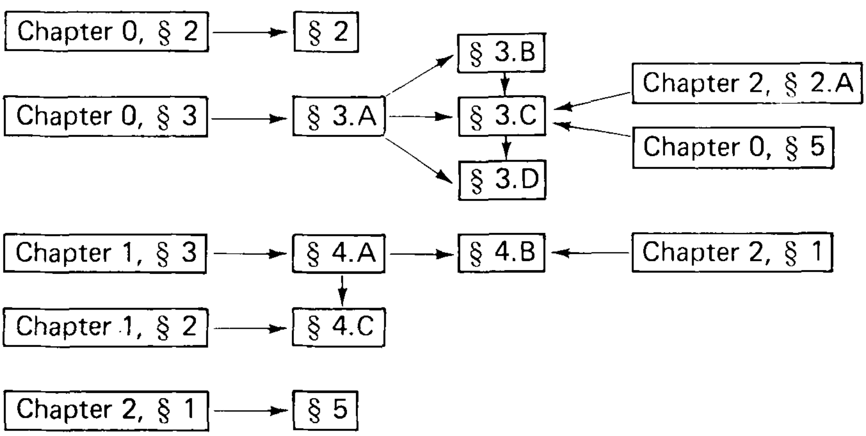
M. A. Dickmann
Paris, April 1974

CHART OF INTERDEPENDENCIES

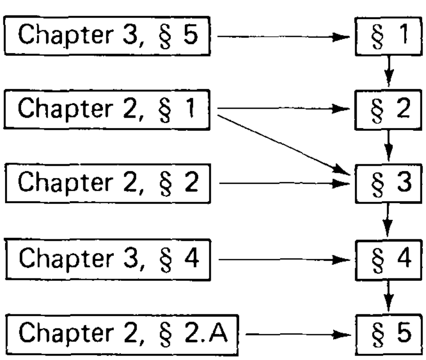
The contents of Chapter 0, § 1 and Chapter 1, § 1 are presupposed and used throughout the book. All the other sections within Chapters 0-2 are conceptually independent from one another, except for the following: In Chapter 0, some concepts and results from §§ 3,4 are used in § 5; the last subsection of Chapter 2, § 2.C, uses some definitions from Chapter 1, § 1.

The logical interdependencies for the rest of the book are indicated, by chapter and section, in the following chart. The sections under consideration appear in the central columns.

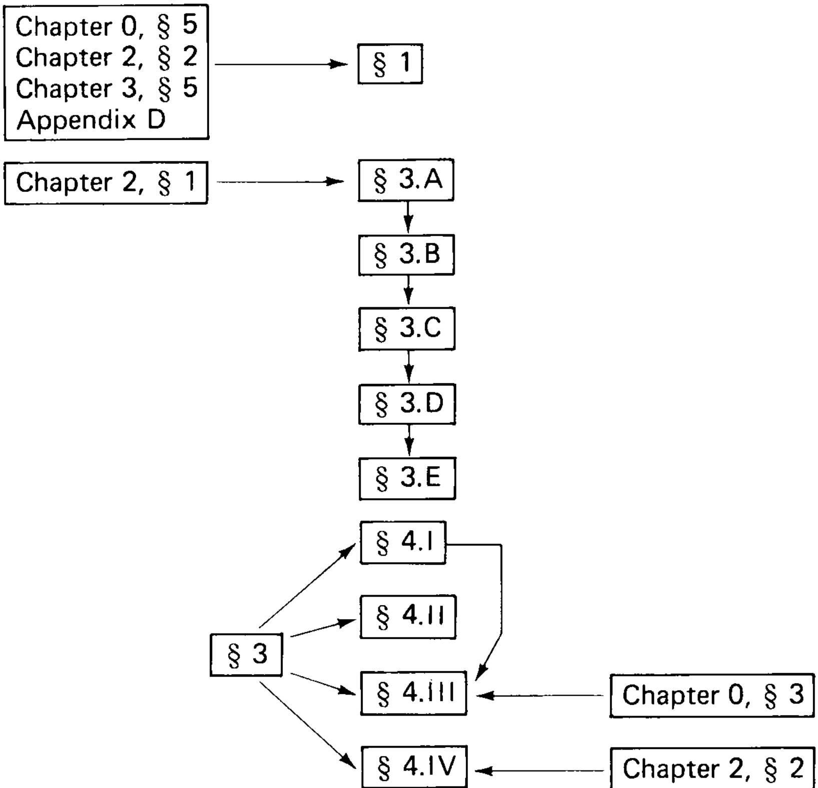
Chapter 3



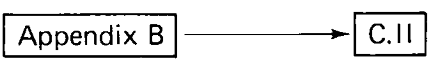
Chapter 4



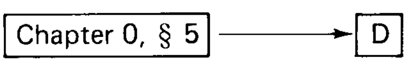
Chapter 5



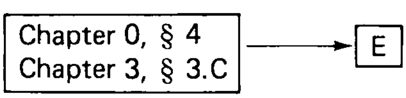
Appendix C



Appendix D



Appendix E



CHAPTER 0

PRELIMINARIES

§ 1. Set-theoretical background

Throughout the greater part of the present book, the set-theoretical framework in which our considerations take place may very well be any of the standard systems of set theory, for instance Zermelo–Fraenkel's set theory (**ZF**), including the axiom of choice (**ZFC**).

In most cases, when we talk about a particular *class*, we just mean a *formula* of the language of **ZF**. Thus, we speak about the class of all ordinals, of all accessible cardinals, of all pairs of natural numbers, etc. All sets defined by a formula of **ZF** (the set of all natural numbers, for instance) are classes. It should be obvious in each case which is the particular formula involved.

The only exception occurs later in these Preliminaries and also in Chapter 3, § 2, where we use ordinals to enumerate the class of all ordinal numbers in the sequence determined by an arbitrary well-ordering. For this, the use of a class-type set theory like Von Neumann–Bernays–Gödel's (**VBG**) will do little good, and the same can be said of the corresponding extensions obtained by considering classes of classes, classes of classes of classes, etc., any finite number of times.

Some of these arguments can be carried out in the (impredicative) set-class system **MK** of Morse–Kelley (cf. KELLEY [1], Appendix). The simplest alternative is, however, to extend **ZFC** by some axiom that guarantees the existence of very large inaccessible cardinals, and assume that the ordinals and cardinals under consideration are all less than some fixed inaccessible number. Such a number should be chosen very large indeed in order to ensure that our statements are not vacuous. Some facts concerning the metamathematical status of such axioms are included below.

Following our friend G. Reyes' (cf. [1], p. 103) famous motto: "Do not scratch if it doesn't itch", we shall be rather brief and sketchy concerning the set-theoretical information used in this book. Any of the numerous good books on set-theory (such as BACHMANN [1], KURATOWSKI–MOSTOWSKI [1] or SIERPIŃSKI [1]) should supply the reader with what is missing here.

We use the standard definition of ordinals à la Von Neumann, whereby an ordinal coincides with the set of all smaller ordinals. In presence of the axiom of foundation (or regularity) an ordinal can be defined as a *transitive set*[1] on which the membership relation $\in$ *is connected*.[2] Thus the order relation between ordinals coincides with the $\in$-relation, or, what is the same, with the relation "$\subset$" of proper inclusion.[3] When there is no possibility of confusion we shall use the symbols "$\in$" or "$<$" interchangeably to denote this relation. **ON** shall denote the class of all ordinal numbers. Lower-case Greek letters (with the exception of "φ", "ψ" and, possibly, a few others) will be used as variables ranging over ordinals.

Given a well-ordered set $\langle x, < \rangle$, there is a unique $\alpha \in \mathbf{ON}$ such that $\langle x, < \rangle \cong \langle \alpha, \in \restriction \alpha \rangle$[4]; such ordinal α is called the *order type* of the set $\langle x, < \rangle$; it is denoted by $\overline{\langle x, < \rangle}$, or simply by $\bar{x}$ when no confusion is possible.

The process that we have just described can be generalized.

DEFINITION 0.1.1. Let K be a class and R a binary relation on K.

(a) R is *extensional* iff for every $x,y \in K$ the following holds:

$$(\text{for every } u \in K,\ uRx \Leftrightarrow uRy) \Rightarrow x = y.$$

(b) R is *well-founded* iff

(i) each non-empty subset of K has an R-minimal element;

(ii) for each $a \in K$, the "R-initial segment" $\{x \in K \mid xRa\}$ is a set.

[1] A class S is *transitive* iff for all x: $x \in S \Rightarrow x \subseteq S$.

[2] A binary relation $R \subseteq S \times S$ is *connected* iff for all $x,y \in S$: $xRy \vee yRx \vee x = y$. Recall that the *domain* of a binary relation $R \subseteq S \times S$ is the set $\{x \in S \mid (\exists y)(\langle x,y \rangle \in R)\}$, the *range* of R is the set $\{y \in S \mid (\exists x)(\langle x,y \rangle \in R\}$, and the *field* of R is the union of its domain and its range.

[3] We shall always use "$\subseteq$" for inclusion and "$\subset$" for proper inclusion.

[4] If R is an n-ary relation on a class $X, R \subseteq X^n$, and $Y \subseteq X$, then $R \restriction Y$ denotes the relation $R \bigcap Y^n$ on Y.

If K_1, K_2 are classes and R_1, R_2 relations on K_1, K_2, respectively, with the same number of arguments, the symbol "$\cong$" denotes the relation of *isomorphism* between $\langle K_1, R_1 \rangle$ and $\langle K_2, R_2 \rangle$.

REMARK. If K is a set, (ii) is superfluous. It is easy to show, using the axiom of choice, that (i) is equivalent to:

(i′) K does not have infinite R-descending sequences, i.e., there is no $f: \omega \to K$[1] such that f is one-one and $f(n+1) R f(n)$ for every $n \in \omega$.

(i′) will be used several times in this book.

The fundamental facts about well-founded relations are:

(1) (Induction principle) Suppose R is a well-founded relation on K and Φ a formula. If for every $x \in K$ we have:

$$(\text{for every } u \in K,\ uRx \Rightarrow \Phi(u)) \Rightarrow \Phi(x),$$

then $\Phi(x)$ holds for every $x \in K$.

Corresponding to (1) there is an obvious principle of definition of operations on K by "induction" on R.

(2) (The Shepherdson–Mostowski theorem) If R is a well-founded and extensional relation on K, there is a *transitive* class M such that $\langle K, R \rangle \cong \langle M, \in \restriction M \rangle$. Moreover, there is a unique isomorphism defined inductively by the condition:

$$F(x) = \{ F(u) \mid uRx \} \quad \text{for } x \in K.$$

This map is called the *collapsing isomorphism* of $\langle K, R \rangle$ onto the transitive class M. A proof of the two last results is given in Appendix A; they will often be used in this book.

An *initial ordinal* is an ordinal α which cannot be put into a one-one correspondence with an ordinal smaller than α. These are the finite ordinals and the "alephs" (or "omegas"). We shall identify *cardinals* with initial ordinals. It is easy to prove from the axiom of choice that for any set x there is a unique initial ordinal which can be put into one–one correspondence with x. This is denoted by $\overline{\overline{x}}$ and called the *cardinal number of* x. **CN** shall denote the class of all infinite cardinals and **CN**′ the class of all cardinals.

There is a map—usually denoted by ω or $\aleph$—such that

(1) $\omega: \mathbf{ON} \to \mathbf{CN}$.

(2) ω is one-one and onto.

[1] Throughout this book we shall use the following notation for maps. Dom(f), Range(f) denote the *domain* and the *range* of f, respectively. $f(x)$ will always denote the *image* of $x \in$ Dom(f) by f. $f[X]$, $f^{-1}[Y]$ (square brackets) shall always denote the *image set* (resp., *inverse image set*) of $X \subseteq$ Dom(f) (resp. of Y) by f. $f: A \to B$ will always mean that Dom(f) $= A$, Range(f) $\subseteq B$.

(3) ω is strictly increasing:

$$\alpha,\beta \in \mathbf{ON} \wedge \alpha \in \beta \Rightarrow \omega_\alpha \in \omega_\beta.$$

(4) ω is continuous:

$$\alpha,\beta \in \mathbf{ON} \wedge \{\gamma_\xi | \xi < \beta\} \subseteq \alpha \wedge \alpha = \bigcup_{\xi<\beta} \gamma_\xi \Rightarrow \omega_\alpha = \bigcup_{\xi<\beta} \omega_{\gamma_\xi}.$$

NOTE. "$\bigcup$" stands for the usual set-theoretical union. If $a \subseteq \mathbf{ON}$, $\bigcup a$ coincides with the "supremum" (least upper bound) or "limit" of the set a.

The number ω_0 coincides with the order type of the set of all natural numbers with their usual ordering. It is also denoted by ω.

NOTATION. If $\{x_i | i \in I\}$ is a family of sets indexed by the set I, $\Pi_{i\in I} x_i$ denotes the set of all maps $f: I \to \bigcup_{i\in I} x_i$ such that $f(i) \in x_i$ for every $i \in I$. If $x_i = x$ for all $i \in I$, $\Pi_{i\in I} x_i$ is just the set of all maps $f: I \to x$ and is denoted by x^I. If $\overline{\overline{x}}_i = \kappa_i$, $\overline{\overline{\Pi_{i\in I} x_i}}$ is denoted by $\Pi_{i\in I}\kappa_i$. If, moreover, the x_i's are pairwise disjoint, then $\overline{\overline{\bigcup_{i\in I} x_i}}$ is denoted by $\Sigma_{i\in I}\kappa_i$. If $\overline{\overline{x}} = \kappa$ and $\overline{\overline{y}} = \lambda$, κ^λ denotes $\overline{\overline{x^y}}$. The existence of most of these sets requires the axiom of choice. $\mathcal{P}(x)$ denotes the power set of x, and $\overline{\overline{\mathcal{P}(x)}}$ is denoted by $2^{\overline{\overline{x}}}$, this notation being consistent with the preceding one. Given an infinite cardinal λ, $\mathcal{P}_\lambda(x)$ shall denote the set of all subsets of x of power $<\lambda$; thus, $\mathcal{P}_\omega(x)$ is the set of all finite subsets of x.

DEFINITION 0.1.2. (a) $\alpha \in \mathbf{ON}$ is a *successor ordinal* iff there is $\beta \in \mathbf{ON}$ such that $\alpha = \beta + 1$ $(= \beta \cup \{\beta\})$; otherwise α is called a *limit ordinal.* If $\alpha = \beta + 1$, then β is denoted by $\alpha - 1$. α is a *limit ordinal* iff $\alpha = \bigcup \alpha$.

(b) $\alpha \in \mathbf{CN}$ is a *successor cardinal* iff it is of the form $\omega_{\beta+1}$; otherwise it is a *limit cardinal.* If $\kappa = \omega_\beta$ and $\alpha = \omega_{\beta+1}$ we write $\alpha = \kappa^+$. If $\alpha = \kappa^+$, then we shall write $\alpha^- = \kappa$; if α is a limit cardinal, $\alpha^- = \alpha$.

(c) If α is a limit ordinal >0, $\mathrm{cf}(\alpha)$ is defined as the smallest $\beta \in \mathbf{ON}$ such that there is a sequence $\{\gamma_\xi | \xi < \beta\} \subseteq \alpha$ such that $\alpha = \bigcup_{\xi<\beta}\gamma_\xi$. $\mathrm{cf}(\alpha)$ is called the *cofinality of* α.

(d) $\kappa \in \mathbf{CN}$ is *singular* iff $\mathrm{cf}(\kappa) < \kappa$. Otherwise, i.e. when $\mathrm{cf}(\kappa) = \kappa$, κ is called *regular*. "**Sing**" and "**Reg**" denote the classes of all singular and regular cardinals, respectively.

(e) $\kappa \in \mathbf{CN}$ is *strongly accessible* iff $\kappa \in$ **Sing** or κ is a successor cardinal. $\kappa \in \mathbf{CN}$ is *accessible* iff $\kappa \in$ **Sing** or $\kappa \leqslant 2^\lambda$ for some cardinal $\lambda < \kappa$. **AC** and

SAC denote the classes of accessible and strongly accessible cardinals, respectively.

(f) $\mathbf{In} = \mathbf{CN} - \mathbf{AC}$; $\mathbf{WIn} = \mathbf{CN} - \mathbf{SAC}$. Elements of **In** are called *inaccessible* or *strongly inaccessible* cardinals; elements of **WIn** are called *weakly inaccessible* cardinals. We rephrase the definitions of **In** and **WIn**:

$\kappa \in \mathbf{WIn} \Leftrightarrow \kappa \in \mathbf{Reg}$ and κ is a limit cardinal;
$\kappa \in \mathbf{In} \Leftrightarrow \kappa \in \mathbf{Reg}$ and for all cardinals $\lambda < \kappa$ we have $2^{\lambda} < \kappa$.[1]

(g) Given $\kappa \in \mathbf{CN}$ we define

$$\beth_0(\kappa) = \kappa,$$

$$\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)},$$

$$\beth_\lambda(\kappa) = \bigcup_{\xi < \lambda} \beth_\xi(\kappa) \quad \text{if } \lambda \text{ is a limit ordinal.}$$

$\beth_\alpha$ denotes $\beth_\alpha(\omega)$; $\beth_\alpha$ is called *beth-α*.

(h) Given $\kappa, \lambda \in \mathbf{CN}$, the cardinal

$$\sum_{\substack{\nu \in \mathbf{CN}' \\ \nu < \lambda}} \kappa^\nu$$

is denoted by $\kappa^{<\lambda}$ and this operation called *weak exponentiation*. We also introduce the operation:

$$\kappa^{>\lambda} = \sum_{\substack{\nu < \lambda \\ \nu \in \mathbf{CN}'}} (\kappa^\nu)^+,$$

which will be called *semi-weak exponentiation*. With obvious modifications these definitions (and the properties stated below) also apply when κ is a finite cardinal $\geqslant 2$.

(i) $\pi : \mathbf{ON} \to \mathbf{ON}$ is defined by the equality

$$\beth_\alpha = \aleph_{\pi(\alpha)}.$$

(By the axiom of choice, π is well-defined.)

In what follows we list—without proofs—some properties that will be needed along these notes; their proofs can be found in any of the aforementioned books on set theory; cf. especially BACHMANN [1], Chapters V–VII.

[1] Cardinals fulfilling the last condition are sometimes called *strong limit cardinals*.

(1) For every limit ordinal $\alpha > 0$, $\mathrm{cf}(\alpha)$ is a regular cardinal, and $\mathrm{cf}(\alpha) \leqslant \alpha$.

(2) For every $\kappa \in \mathbf{CN}$, $\mathrm{cf}(\kappa^+) = \kappa^+$, i.e. $\kappa^+ \in \mathbf{Reg}$.

(3) π is strictly increasing and continuous; hence $\pi(\gamma+1) > \gamma$ or $\beth_{\gamma+1} > \omega_\gamma$ for all $\gamma \in \mathbf{ON}$.

(4) $\alpha \in \mathbf{ON} \Rightarrow \mathrm{cf}(\omega_\alpha) \leqslant \omega_\alpha \leqslant \omega_{\pi(\alpha)}$.

(5) König's theorem: $\Sigma_{i \in I} \kappa_i < \Pi_{i \in I} \lambda_i$ whenever $\kappa_i < \lambda_i$ for all $i \in I$.

(6) $cf(\beth_{\alpha+1}) > \omega_\alpha$ and $cf(2^\lambda) > \lambda$.

(7) If ω is cofinal in α then $2^{\aleph_0} \neq \aleph_\alpha$. In particular $2^{\aleph_0}$ is different from $\aleph_\omega$, $\aleph_{\omega_\omega}$, $\aleph_{\omega+\omega}$, etc.

(8) $\lambda^{\mathrm{cf}(\lambda)} > \lambda$.

(9) $\lambda^\kappa = \lambda \Rightarrow \kappa < \mathrm{cf}(\lambda)$.

(10) $\lambda^{<\kappa} = \lambda \Rightarrow \kappa \leqslant \mathrm{cf}(\lambda)$.

(11) (i) α successor ordinal:

$$\beth_\alpha^\kappa = \begin{cases} \beth_\alpha & \text{if } \kappa \leqslant \beth_{\alpha-1}, \\ 2^\kappa & \text{if } \kappa > \beth_{\alpha-1}. \end{cases}$$

(ii) α limit ordinal:

$$\beth_\alpha^\kappa = \begin{cases} \beth_\alpha & \text{if } \kappa < \mathrm{cf}(\beth_\alpha)(=cf(\alpha)), \\ \beth_{\alpha+1} & \text{if } \mathrm{cf}(\alpha) \leqslant \kappa \leqslant \beth_\alpha, \\ 2^\kappa & \text{if } \kappa > \beth_\alpha. \end{cases}$$

(12)

$$(\kappa^{<\lambda})^\rho = \begin{cases} \kappa^{<\lambda} & \text{if } \rho < \mathrm{cf}(\lambda), \\ \kappa^\lambda & \text{if } \mathrm{cf}(\lambda) \leqslant \rho \leqslant \lambda, \\ \kappa^\rho & \text{if } \rho \geqslant \lambda. \end{cases}$$

(13)

$$(\kappa^{<\lambda})^{<\rho} = \begin{cases} \kappa^{<\lambda} & \text{if } \rho \leqslant \mathrm{cf}(\lambda), \\ \kappa^\lambda & \text{if } \mathrm{cf}(\lambda) < \rho \leqslant \lambda^+, \\ \kappa^{<\rho} & \text{if } \rho \geqslant \lambda^+. \end{cases}$$

(14)

$$(\kappa^\lambda)^{<\rho} = \begin{cases} \kappa^\lambda & \text{if } \rho \leqslant \lambda^+, \\ \kappa^{<\rho} & \text{if } \rho \geqslant \lambda^+. \end{cases}$$

(15) κ strong limit cardinal $\Leftrightarrow 2^{<\kappa} = \kappa = 2^{>\kappa}$.

(16) $\alpha > 0 \Rightarrow (\omega_\alpha \in \mathbf{WIn} \Leftrightarrow \alpha = \omega_\alpha = \mathrm{cf}(\alpha))$.
(17) $\omega \in \mathbf{In} \subseteq \mathbf{WIn}$ (hence $\mathbf{SAC} \subseteq \mathbf{AC}$).
(18) $\omega_\alpha \in \mathbf{In} \Leftrightarrow \omega_\alpha \in \mathbf{WIn} \wedge \pi(\alpha) = \alpha$.
(19) $\kappa \in \mathbf{In} \Leftrightarrow \kappa$ is a strong limit cardinal and $\kappa^{<\kappa} = \kappa \Leftrightarrow$ for every collection $\{\kappa_i | i \in I\}$ of cardinals such that $\bar{\bar{I}} < \kappa$ and $\kappa_i < \kappa$ for all $i \in I$, $\Pi_{i \in I} \kappa_i < \kappa$.
(20) $\kappa^{<\lambda} \leqslant \kappa^{>\lambda} \leqslant (\kappa^{<\lambda})^+$.
(21) If λ is not a strong limit cardinal (e.g. if λ is a successor cardinal), then $\kappa^{>\lambda} > \lambda$.
(22) If $\kappa^{>\lambda} > \lambda$, then $(\kappa^{<\lambda})^+$ is the smallest regular cardinal bigger or equal than $\kappa^{>\lambda}$; in particular:
(23) If $\kappa^{>\lambda}$ is regular and $\kappa^{>\lambda} > \lambda$, then $\kappa^{>\lambda} = (\kappa^{<\lambda})^+$; if $\kappa^{>\lambda}$ is singular or $\kappa^{>\lambda} \leqslant \lambda$, then $\kappa^{>\lambda} = \kappa^{<\lambda}$.

The *generalized continuum hypothesis* (**GCH**) is the following statement:

$$\text{for all } \kappa \in \mathbf{CN}: \qquad 2^\kappa = \kappa^+.$$

The **GCH** has many important consequences in the arithmetic of cardinal numbers; we list some refinements of preceding results *assuming the* **GCH**:

(24) For all $\alpha \in \mathbf{ON} : \pi(\alpha) = \alpha$ (i.e., $\beth_\alpha = \aleph_\alpha$).
(25) $\mathbf{In} = \mathbf{WIn}$.
(26) (11) holds with $\aleph_\alpha$ instead of $\beth_\alpha$.
(27) $\lambda^{<\mathrm{cf}(\lambda)} = \lambda$.
(28) $\kappa \leqslant \mathrm{cf}(\lambda) \Leftrightarrow \lambda^{<\kappa} = \lambda$.
(29) $\kappa < \mathrm{cf}(\lambda) \Leftrightarrow \lambda^\kappa = \lambda$.
(30)

$$\lambda^{<\lambda} = \begin{cases} \lambda & \text{if } \lambda \text{ regular,} \\ \lambda^+ & \text{if } \lambda \text{ singular.} \end{cases}$$

(31) $2^{<\kappa} = \kappa$.

Gödel proved in 1938 (cf. [1]) that the **GCH** is consistent with **ZF**. Cohen proved in 1963 that its negation is consistent with **ZFC**.

Sometimes it is possible to dispense with the **GCH** in favor of weaker assumptions. For example, (25) is derivable from the so-called *limit cardinal hypothesis* (**LCH**):

"every limit cardinal is a strong limit cardinal" or, in other words:

if λ is a limit cardinal and $\kappa<\lambda$, then $2^{\kappa}<\lambda$.

A set-theoretical assumption weaker than the **GCH** is used, for example, in Theorem 0.4.23 below.

We recall in passing the definition of the *ordinal arithmetical operations*, operations which ocassionally will be used in this book.

DEFINITION 0.1.3. Let $\langle\alpha_\xi\,|\,\xi<\lambda\rangle$ be a well-ordered sequence of ordinals.

(a) *Addition*. Let $\langle\langle X_\xi, <_\xi\rangle\,|\,\xi<\lambda\rangle$ be a sequence of pairwise disjoint well-ordered sets such that $\overline{\langle X_\xi, <_\xi\rangle}=\alpha_\xi$. We define a binary relation $\prec$ on $\bigcup_{\xi<\lambda}X_\xi$ as follows:

$$a \prec b \Leftrightarrow (a\in X_\xi\wedge b\in X_\eta\wedge\xi<\eta)\vee(a\in X_\xi\wedge b\in X_\xi\wedge a<_\xi b),$$

It is easily seen that this relation well-orders $\bigcup_{\xi<\lambda}X_\xi$. We set:

$$\sum_{\xi<\lambda}\alpha_\xi=\overline{\left\langle\bigcup_{\xi<\lambda}X_\xi,\ \prec\right\rangle}$$

(This is a proper definition, i.e., $\Sigma_{\xi<\lambda}\alpha_\xi$ depends only on the ordinals α_ξ, not on the particular sets $\langle X_\xi, <_\xi\rangle$ of that type.)

(b) *Product*. Consider the set $\Pi^*_{\xi<\lambda}\alpha_\xi$ consisting of all maps $f\in\Pi_{\xi<\lambda}\alpha_\xi$ such that $f(\xi)=0$ except for finitely many ξ's. We order it by "last differences": if $f,g\in\Pi^*_{\xi<\lambda}\alpha_\xi$, then

$$f\prec' g \quad\Leftrightarrow\quad f(\xi_0)\in g(\xi_0),\text{ where }\xi_0\text{ denotes the largest element of the set }\{\xi<\lambda\,|\,f(\xi)\neq g(\xi)\}.$$

Again, it is easily proved that $\prec'$ is a well-ordering; we set:

$$\prod_{\xi<\lambda}\alpha_\xi=\overline{\left\langle\prod_{\xi<\lambda}{}^*\alpha_\xi,\ \prec'\right\rangle}\,.$$

As usual, the product of two factors is denoted by $\alpha\cdot\beta$.

(c) As a particular case, if $\alpha_\xi=\alpha$ for all $\xi<\beta$, the product $\Pi_{\xi<\beta}\alpha_\xi$ is called *ordinal exponentiation* and is denoted by α^{β}.

The reader is referred to BACHMANN [1] and SIERPIŃSKI [1] for a full treatment of ordinal arithmetic. Here we limit ourselves to a few remarks that will help to make clear the sense in which the ordinal operations are used in this book.

Note first that $\overline{\overline{\Sigma_{\xi<\lambda}\alpha_\xi}} = \Sigma_{\xi<\lambda}\overline{\overline{\alpha_\xi}}$; in particular, if each α_ξ is a cardinal (i.e. an initial ordinal) and $\xi<\alpha \Rightarrow \alpha_\xi<\alpha_\delta$, it is readily seen that:

$$\left\langle \bigcup_{\xi<\lambda} X_\xi, \prec \right\rangle \cong \langle \kappa, \epsilon \restriction \kappa \rangle$$

where κ is the (cardinal) sum of the cardinals $\alpha_\xi, \xi<\lambda$ (cf. p. 4).

The ordinal product is an iterated sum: if $\alpha, \beta \in \mathbf{ON}$ and $\alpha_\xi=\alpha$ for all $\xi<\beta$, then $\alpha\cdot\beta=\Sigma_{\xi<\beta}\alpha_\xi$. Hence, if α, β are cardinals, so is $\alpha\cdot\beta$ and $\langle\alpha\cdot\beta, \prec\rangle\cong\langle\alpha\cdot\beta, \in\restriction(\alpha\cdot\beta)\rangle$; in other words, the ordinal product of finitely many initial ordinals coincides with its cardinal product. This is not true for infinitely many factors and, in particular, for exponentiation; e.g., the ordinal exponentiation ω^ω is a denumerable ordinal, whereas the corresponding cardinal operation has $2^{\aleph_0}$ as result.

In the few cases where confusion may result from the use of identical notation for both the cardinal and ordinal operations, we will attach the word "ordinal" to the last; for example, "ω^ω (ordinal exponentiation)" denotes a denumerable ordinal, while "ω^ω" denotes the cardinal $2^{\aleph_0}$. Whenever the notation "$\aleph$" is used, all the operations are cardinal.

A number of laws valid in the elementary arithmetic of natural or real numbers hold for the ordinal arithmetical operations as well. For example, we mention the associative laws for addition and product, the left-distributivity of addition with respect to multiplication ($\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$), some exponential identities (e.g., $\alpha^{\beta+\gamma}=\alpha^\beta\cdot\alpha^\gamma$, $(\alpha^\beta)^\gamma=\alpha^{\beta\cdot\gamma}$), right-monotonicity of both addition and multiplication ($\beta<\gamma$ and $\alpha\neq0$ imply $\alpha+\beta<\alpha+\gamma$ and $\alpha\cdot\beta<\alpha\cdot\gamma$), right-continuity of these operations (i.e., $\lim_{\xi<\lambda}(\alpha+\beta_\xi)=\alpha+\lim_{\xi<\lambda}\beta_\xi$; $\lim_{\xi<\lambda}(\alpha\cdot\beta_\xi)=\alpha\cdot(\lim_{\xi<\lambda}\beta_\xi)$), continuity of exponentiation as a function of the exponent, etc.

However, many well-known arithmetical statements do not extend to ordinal arithmetic. For instance, sum and product are not commutative (e.g., $1+\omega<\omega+1$, $2\cdot\omega=\omega<\omega\cdot2$), they are not monotonous on the left (e.g., $2\cdot\omega=3\cdot\omega$, $2+\omega=3+\omega$), they are not left-continuous (e.g., $\lim_{n<\omega}(n+\omega)=\omega<\omega+\omega=(\lim_{n<\omega}n)+\omega$), exponentiation does not distribute over product (e.g., $(2\cdot2)^\omega<2^\omega\cdot2^\omega$), etc.

The sets $R(\alpha)$ are inductively defined as follows:

$$R(0)=\emptyset$$

$$R(\alpha)=\bigcup_{\beta<\alpha}\mathcal{P}(R(\beta)) \quad \text{for } \alpha>0.$$

Let $\Pi = \bigcup_{\alpha \in \mathbf{ON}} R(\alpha)$. It is easy to prove that the axiom of foundation is equivalent to the statement $\Pi = \mathrm{V}$. (V denotes the class of all sets.) Therefore every set belongs to some $R(\alpha)$. The rank of a set x, denoted by $\rho(x)$, is the least $\alpha \in \mathbf{ON}$ such that $x \in R(\alpha+1)$. The following properties can be easily proved by the reader:

(1) $\alpha \leqslant \beta \Leftrightarrow R(\alpha) \subseteq R(\beta)$.
(2) $\alpha \in \mathbf{ON} \Rightarrow \rho(\alpha) = \alpha$.
(3) $x \in y \Rightarrow \rho(x) < \rho(y)$.
(4) $\rho(x) = \bigcup \{\rho(y)+1 \mid y \in x\}$.
(5) $x \in R(\beta) \Leftrightarrow \rho(x) < \beta$.
(6) $(x \in y \vee x \subseteq y) \wedge y \in R(\beta) \Rightarrow x \in R(\beta)$.
(7) $a \subseteq \mathbf{ON} \Rightarrow \bigcup_{\beta \in a} R(\beta) = R(\gamma)$ for some $\gamma \in \mathbf{ON}$.

The "natural" structures are defined as:

$$\mathfrak{B}_\alpha = \langle R(\alpha), \in \restriction R(\alpha) \rangle.$$

It is not difficult to prove that
(A) $\mathfrak{B}_{\alpha+1}$ is a model of **VBG** $\Leftrightarrow \alpha \in \mathbf{In} - \{\omega\}$,
(B) $\alpha \in \mathbf{In} - \{\omega\} \Rightarrow \mathfrak{B}_\alpha$ is a model of **ZF**.
The converse of (B) is not true as shown by MONTAGUE–VAUGHT [1].

The results that we have just quoted are the basis for showing that the statement "$\mathbf{In} = \{\omega\}$" is consistent with the axioms of set theory, **ZF** say. If $\mathbf{In} - \{\omega\} \neq \emptyset$, let α_0 be its first element. By (B), $\mathfrak{B}_{\alpha_0}$ is a model of **ZF**; it is not difficult to show that $\mathbf{In}^{\mathfrak{B}_{\alpha_0}} = \{\omega\}$.[1]

The same idea can be used to show that if the statement "$\mathbf{In} - \{\omega\} \neq \emptyset$" is consistent with **ZF** (**ZFC**), then the following statement is also consistent with **ZF** (**ZFC**):

"there is exactly one $\alpha \in \mathbf{In}$, $\alpha > \omega$".

Analogous results can be proved replacing "one" by "two", "three", etc. Hence *each existential assumption about inaccessibles $> \omega$ requires a particular axiom.*

Finally, it should be remarked that *it is not known whether* "$\mathbf{In} - \{\omega\} \neq \emptyset$" *is consistent with* **ZF**.

It is convenient to make at this point a reference to the class L of Gödel and a variant of it, L_a, which will be mentioned—though not used directly—in this book. For details we refer the reader to GÖDEL [1].

[1] If X is a class and $\mathfrak{M}$ is a model of **ZF** (**ZFC**), $X^{\mathfrak{M}}$ denotes the class corresponding to X in the sense of the model $\mathfrak{M}$.

The definition of the classes L and L_a (for a fixed set a) is done by transfinite induction; $L(\alpha+1)$ consists of all subsets of $L(\alpha)$ definable (in $\langle L(\alpha), \in \restriction L(\alpha)\rangle$) by a formula involving $\in$ using parameters from $L(\alpha)$; for limit ordinals α, $L(\alpha) = \bigcup_{\beta<\alpha} L(\beta)$. $L_a(\alpha+1)$ is analogous, except that the formula involves an additional unary predicate; likewise, $L_a(\alpha) = \bigcup_{\beta<\alpha} L_a(\beta)$ for limit ordinals α. The elements of L are called *constructible sets* and L itself the *constructible universe*.

We need only know the following properties:

(1) L and L_a are transitive models of **ZFC** containing all ordinals.

(2) $a \cap L_a \in L_a$ (but not necessarily $a \in L_a$!).

(3) If $\mathfrak{M}$ is a transitive model for **ZF** including **ON**, then $L \subseteq \mathfrak{M}$ (hence $L \subseteq L_a$).

(4) If $\mathfrak{M}$ is a transitive model for **ZF** including **ON** and such that $a \cap \mathfrak{M} \in \mathfrak{M}$, then $L_a \subseteq \mathfrak{M}$ (hence if $a \in L$ then $L_a = L$).

Since L is a class, $V = L$ is a meaningful statement (called the *axiom of constructibility*); same for $V = L_a$.

(5) $L^L = L$; therefore $V = L$ holds in L. Similarly, $V = L_a$ holds in L_a.

The importance of $V = L$ comes from the following:

(6) $V = L \Rightarrow$ **GCH**. Thus from (5) it follows that **GCH** is consistent with **ZFC**.

§ 2. "Constructibly" defined inaccessible cardinals

From the information contained in the preceding section it is clear that inaccessible numbers $>\omega$ are rather large. However, we will go several steps further in the classification of large cardinals.

It should be obvious that all arguments involving inaccessible cardinals are highly speculative, *since we do not have at present a consistency proof of their existence.*

In order to make meaningful our considerations, it is advisable to introduce very strong axioms concerning the existence of inaccessibles. Some examples are:

> "**In** is cofinal in **ON**" (Tarski),

or

> "Every strictly increasing, continuous function f: **ON**→**ON** has at least one inaccessible number in its range" (Lévy),

or even stronger ones.

Since we want to keep our discussion at an informal level, we leave to the reader to figure out what axioms are necessary in each case.