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Complex Polynomials

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This book studies the geometric theory of polynomials and rational functions in the plane. Any theory in the plane should make full use of the complex numbers and thus the early chapters build the foundations of complex variable theory, melding together ideas from algebra, topology and analysis.

In fact, throughout the book, the author introduces a variety of ideas and constructs theories around them, incorporating much of the classical theory of polynomials as he proceeds. These ideas are used to study a number of unsolved problems, bearing in mind that such problems indicate the current limitations of our knowledge and present challenges for the future. However, theories also lead to solutions of some problems and several such solutions are given including a comprehensive account of the geometric convolution theory.

This is an ideal reference for graduate students and researchers working in this area.

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Preface

Polynomials play an important role in almost all areas of mathematics. From finding solutions to equations, finding the number of solutions of equations, understanding the role of critical points in determining the geometric behaviour of the distribution of values, determining the properties of geometric curves and much else, polynomials have wielded an enormous influence on the development of mathematics since ancient times. This is a book on polynomials in the plane with a special emphasis on the geometric theory. We can do justice to only a very small part of the subject and therefore we confine most of our attention to the study of a number of specific problems (some solved and some unsolved). However, as the book is directed towards graduate students and to a broad audience of scientists and mathematicians possessing a basic knowledge of complex variable theory, we have concentrated in the earlier chapters on building some theoretical foundations, melding together algebraic, topological and analytic ideas.

Description of the chapters.

Chapter 1. **The algebra of polynomials.** The chapter deals with two important issues concerning real analytic polynomials – a topic normally regarded as being part of classical algebraic geometry. The first is a straightforward account of Bézout's theorem in the plane. We prove a version of the theorem which is independent of any algebraic restrictions, and therefore immediately applicable and useful to the analyst. The second topic concerns the calculation and properties of asymptotic values at infinity. We give details of an algorithm, with which one can calculate all asymptotic values and the tracts within which they lie. The method is not quite computable, since it involves finding the exact zeros of certain polynomials. However, we establish two important facts: (1) all finite limiting values at infinity are asymptotic values; (2) the repetition property of asymptotic values. This work will be applied in our study of the Jacobian problem in chapter 3.

Chapter 2. **The degree principle and the fundamental theorem of algebra.** The first half of this chapter is effectively a mini-course in plane topology, and in fact is based on undergraduate lectures given at York c. 1980. The degree principle, as stated and proved, is a fundamental rotation theorem in plane topology and deserves to be as familiar to all mathematicians as is Cauchy's theorem, to which it is closely analogous. Its corollary, the argument principle, is a powerful general result for locating certain types of 'singularity' of continuous functions and immediately places the classical argument principle for analytic or meromorphic functions as a topological rather than an analytic result. The statement of the principle is virtually unchanged from the classical statement, providing only that one has a clear concept of topological multiplicity. Some theorems, such as Rouché's theorem, have topological analogues, which include the Brouwer fixed point theorem. We are also able to prove, not only the fundamental theorem of algebra, but existence of a minimum number of zeros for more general polynomial-type functions. Included here are the complex-valued harmonic polynomials (with a mild restriction). We show that a harmonic polynomial of degree n has at least n zeros and, using the algebra of chapter 1, at most n^2 zeros; we include Wilmshurst's elegant example of a harmonic polynomial with exactly n^2 zeros. We also begin in this chapter our study of locally 1–1 mappings of the plane, including the important theorem that the global injectivity depends entirely on the behaviour at infinity. We introduce a second topological method 'continuation of the inverse function' using the monodromy theorem. The final part of the chapter contains a detailed study of harmonic polynomials on the critical set.

Chapter 3. **The Jacobian problem.** The problem of when it is possible to solve uniquely a set of simultaneous algebraic equations is among the most beautiful problems of classical mathematics and remains only partially resolved. This chapter is a 'bottom up' and elementary account of the two-dimensional case, in which we consider the sufficiency of the non-vanishing of the Jacobian. In line with chapters 1 and 2 we adopt a quite specific approach by examining the possibility and nature of a polynomial mapping having asymptotic values at infinity. For real variables the central question has been resolved by S. Pinchuk, who has constructed a polynomial mapping of degree 25 with a positive Jacobian, but which is not globally 1–1. We give the details of his elegant construction and then by explicitly calculating the asymptotic values are able to examine the geometric and topological behaviour of the mapping.

Topologically Pinchuk's example turns out to be as simple as such a mapping can be and lends credence to the possibility that below degree 25 the main conjecture is true (this also seems likely from the simplicity of the original algebraic construction). We consider this question and obtain some estimates for the degree of counter-examples. As the author's algebraic capacity and resources are limited, we suggest that resolving this question is a suitable topic for a PhD research thesis.

For complex variables the Jacobian is constant, a fact which has strong algebraic implications. There is a massive literature on this problem (see e.g. [1], [5], [66]), much of it algebraically very advanced. The conjecture is true up to degree 100 and also under various additional conditions on the degrees of the components of the mapping. The general conjecture is unresolved. We continue with our elementary approach using asymptotic values, obtaining a few of the known results and, we hope, leading the student to a greater understanding of the complexity of the problem and the probable need for more advanced methods. We also include a topological approach to the problem, which certainly turns up some interesting and strange features.

Chapter 4. **Analytic and harmonic functions in the unit disc.** In this chapter we begin our study of convolutions and convolution operators, which is one of the central themes of the book. A knowledge of general real analysis and the theory of Fourier series will be useful here. The main themes are the annihilation properties of convolution operators, the representation of general linear operators, the structure preserving properties of bounded and positive operators and the construction of approximate identities. We also introduce a class of multiplication operators and use these to give very simple proofs of some classical polynomial inequalities, which we are able to generalise. We include an account of variation diminishing transformations and give Pólya and Schoenberg's proof that the de la Vallée Poussin means form a variation diminishing approximate identity. The chapter concludes with a brief account of generalised convolution operators, including a coefficient inequality due, essentially, to Rogosinski. As a whole non-trivial class of such operators is established in chapter 8, we have here a family of (unstated) theorems and inequalities.

Chapter 5. **Circular regions and Grace's theorem.** This is our main chapter on the classical theory 'Geometry of the zeros', of which the centrepiece is the beautiful theorem of Grace. Our approach to the theorem and its proof is unorthodox and is, in part, motivated by the need to set the scene for the powerful generalisations of chapter 8. We begin by introducing a general concept of duality defined in terms of convolutions. This is related to the functional analysis concept but is centred around a quite different class of functions: analytic functions with no zeros in the unit disc. Grace's theorem is stated as a second dual theorem and is closely related to G. Szegö's version expressed in terms of linear functionals. The proof given is due to the author, but is based on an ingenious idea due to St. Ruscheweyh, which we have expressed as the 'finite product lemma'. This lemma will form a central plank in the later convolution theory. After transforming Grace's theorem into a theorem for general circular regions we define apolarity and prove Grace's apolarity theorem and Walsh's theorem on symmetric linear forms. We then relate the theory to the more conventional geometric theory, proving a number of inequalities and giving an account of a number of Walsh's geometric theorems, including his celebrated 'two circles theorem'. We also include an account of the recent generalisation of Bernstein's theorem to rational functions due to Borwein and Erdélyi. At the expense of a slightly weaker result, we give a quite different proof using the methods of this chapter and also obtaining some other inequalities. We then establish the circle of theorems surrounding the Grace–Heawood theorem including the interesting 'bisector' theorem of Szegö. The chapter concludes with an extension of Grace's theorem for general linear operators.

Chapter 6. **The Ilieff–Sendov conjecture.** This problem, still unsolved, is one of those problems which create a mathematical itch. The question whether, given a polynomial with all its zeros in the unit disc, there is necessarily a critical point within unit distance of a given zero, is an innocent sounding proposition, which one would expect to be able to resolve quite quickly. Indeed Q.I. Rahman, who has contributed some of the most attractive ideas to the problem, once had the experience of mentioning this problem in a lecture attended by engineers, who laughed at the silliness of the question. He therefore offered a prize to any who could solve the problem by the end of the conference. For the remainder of the conference groups of engineers could be seen trying to answer this trivial question. Of course, Rahman kept his money. The problem is now approaching its fortieth birthday. However, let not age deter the bright young newcomer. Simple problems do sometimes have simple solutions, but nevertheless last a long time. The author has twice had the experience of producing simple solutions to old problems – one lasted over forty years, the other seventy-seven! (Both are in this book). However, a word of warning: simple solutions are rarely simply found and can cost a great deal of time and effort. Most (all?) mathematicians spend 99% of their time failing to solve the problems in which they are interested. The author would be delighted with a 1% success rate: the solution to the seventy-seven year old problem reduced to a short paragraph of verbal reasoning, but took eight years to discover.

For the Ilieff–Sendov conjecture, we concentrate on presenting some of the general ideas and methods, which have been obtained. A recent result is in an interesting and informative paper of J.E. Brown and G. Xiang [12], who establish the conjecture for polynomials up to degree 8 and further for polynomials with at most eight distinct zeros.

Chapter 7. **Self-inversive polynomials.** These are polynomials with their zeros either on the unit circle or distributed in pairs conjugate to the circle. After a few preliminaries our main attention is focused on polynomials whose zeros lie on the circle. We establish some inequalities for integral means and coefficients relating these to the maximum modulus on the circle. We also establish an interspersion theorem for the ratio of two polynomials whose zeros are interspersed on the circle. However, the bulk of the chapter is an exposition of T.J. Suffridge's startling theory of polynomials whose zeros lie on the circle and are separated by a minimum specified gap. He shows that such polynomials can be used to approximate some geometrically defined classes of analytic functions in the unit disc. Furthermore the polynomials can be classified according to a number of different criteria. However, his deepest and most significant result is his convolution theorem for such polynomials. This is undoubtedly one of the most beautiful results in the entire theory of polynomials and deserves to be much better known and understood. Unfortunately, because of the length and complexity of the proof, we are able to give only a partial account of his main argument. It would be a useful research project for someone to take apart and, if possible, simplify his methods. We make one suggestion for a possible simplification of one part of his theorem.

Chapter 8. **Duality and an extension of Grace's theorem to rational functions.** This chapter is the heart of our development of the convolution theory and contains the deepest results in the book. The central result is that the Kaplan classes (defined in chapter 7) lie in second dual spaces for very simple classes of functions. Part of the importance of duality theory lies in its connection with convexity and extreme point theory, and we explain this. Furthermore, the Kaplan classes form very general classes of non-vanishing analytic functions in the disc, and therefore the theorems and inequalities obtained give results of considerable generality and power. The theory is a development, taking place over a number of years, of the work of St. Ruscheweyh and the author arising out of their proof of the Pólya–Schoenberg conjecture: that the convolution of two convex univalent mappings of the disc remains convex univalent, and the chapter includes this result. However, we have emphasised in this book the importance to this development of the ideas and results of earlier authors: Szegö, Pólya and Schoenberg, Suffridge. Although the main theorems are not theorems on polynomials, nevertheless they arise naturally out of the polynomial theory, as we show, and in any case include a whole range of results for various types of finite and infinite products. Certain structure preserving classes arise in this theory and identifying these explicitly is a central problem. The author posed to his research student S. Robinson the problem of finding polynomial members. Robinson undertook a substantial amount of computer work and by means of some brilliant pattern spotting proposed certain specific polynomials. Although he had never heard of them, these turned out to be the generating polynomials for the Cesàro means! We have included his conjectures as beautiful unsolved problems.

Chapter 9. **Real polynomials.** This chapter centres on some questions of a quite specific type, namely estimating the number of real, or non-real, zeros or critical points of various types of real polynomials and rational functions. This constitutes a return to topological and geometric thinking typical of chapter 2. After introducing the Jensen circles we give a detailed discussion of the unresolved conjecture of Craven, Csordas and Smith (recently nicknamed the Hawaii conjecture, as the three authors are all from the University of Hawaii), concerned with relating the number of non-real zeros of a real polynomial to the number of real critical points of the logarithmic derivative. As this conjecture relates closely to the topological structure formed by the level curves on which the logarithmic derivative is real, it is a problem of fundamental interest in understanding the structure of real polynomials.

After a discussion of strongly real rational functions (rational functions which are real only on the real axis), we establish a general theorem on critical points. We then generalise this theory to real entire and meromorphic functions, concluding with a proof of Wiman's conjecture relating the genus of a real entire function of finite order possessing only real zeros to the number of non-real zeros of the second derivative. This is an account of the author's proof of the conjecture, in which we have simplified some of the original arguments in order to avoid the use of advanced methods and to make the argument self-contained.

Chapter 10. **Level curves.** Level curves have appeared many times earlier in the book. Here we establish two of the main facts: the relationship between polynomials in level regions and Blaschke products in the unit disc, and for rational functions the important Riemann–Hurwitz formula relating for a level region the number of zeros, critical points and components of the boundary.

The final part of the chapter centres around the fascinating Smales' conjecture, still unsolved after twenty years and originally formulated by S. Smale as a result he would like to be true for its usefulness in the theory of computer complexity. We have included our account in a chapter on level curves, because of the geometric orientation which study of the problem constantly arouses. In particular the conjecture raises questions concerning the relationship between the shape of level curves and the number of critical points (inside or near by). Although Hilbert long ago showed that the level curve of a polynomial could have any shape (specifically, given two Jordan curves, one inside the other thus creating an annular region beween the curves, one can construct a polynomial possessing a level curve lying inside the annular region and containing the inner of the two curves in its interior), nevertheless this says nothing about the critical points. The recent book by Borwein and Erdélyi [9] gives substantial coverage of the metric theory of level curves or 'lemniscates'.

Chapter 11. **Miscellaneous topics.** This chapter contains one extended piece of theory plus some isolated and striking results. Mason's *abc* theorem and its corollary Fermat's last theorem for polynomials were taken from a popular work on mathematics by Ian Stewart [60] simply for their attractiveness. We have expressed the theorem as a particular case of a result giving a lower bound on the number of points at which a rational function can take on specified values. Cohn's lemma is a well-known and useful result on finding the number of zeros of a polynomial in the unit disc. Blaschke products have been prominent in many places in the book. Here we give an extended account of a variety of features. Firstly we include Walsh's beautiful analogue of the Gauss–Lucas theorem and the connection with non-euclidean geometry. We then move on to a detailed account of the connections between Blaschke products and certain harmonic mappings of the disc, particularly the harmonic extension into the disc of a step function on the circle, of mappings onto convex curves and of mappings onto polygons. Included here are a proof of a conjecture of H.S. Shapiro on the Fourier coefficients of an n-fold mapping of the circle, a proof of the Rado–Kneser–Choquet theorem on the univalence of the harmonic extension of a mapping of the circle onto a convex curve and a proof of the finite valence of the analytic part of the harmonic extension for n-fold mappings. Also included is a statement of the as yet unsolved rotation conjecture of the author. The final topic of the book is Sudbery's proof of Popoviciu's conjecture on the number of distinct zeros of the successive derivatives of a polynomial. We extend Sudbery's result by considering the number of zeros of just some of the derivatives.

Concluding remarks. Prerequisites for this book are graduate level real analysis, basic topology and algebra and some complex variable theory, preferably up to and including the Riemann mapping theorem. We have made every effort to avoid advanced material and very technical knowledge in the hope that the work will then be available to a wide audience. We have included a scattering of problems for the reader; however, the book has not been written as a textbook; it is more a presentation of ideas, methods and especially proofs. The interested reader should certainly consult the classic works on polynomials, notably Marden [30], Walsh [69], Pólya and Szegö [41] and Levin [25], as well as more recent works: the book by Borwein and Erdélyi [9] and the forthcoming work of Rahman and Schmeisser cover numerous topics not mentioned in this book, including substantial amounts of work connected with approximation theory. Iteration theory is nowadays an important subject in its own right and is very fully covered in a number of works. Finally, the author has uncovered many errors in his various readings of the text. Inevitably many will remain undetected. The author apologises for these and emphasises that such errors are the author's errors and should not be attributed to the writers being discussed.