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Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups

Second Edition

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ABSTRACT. This is a revised and enlarged edition of the book with the same title published by the Princeton University Press in 1980 which was concerned with various types of cohomology theories pertaining to Lie groups (real or p-adic), Lie algebras, infinite dimensional representations, and to cocompact discrete subgroups of reductive groups. Apart from corrections and minor changes or amplifications, the text of the original edition has been kept. It has been augmented notably by various additions on the Zuckerman functors, the Vogan-Zuckerman theorem describing the relative Lie algebra cohomology with coefficients in an irreducible unitary representation, and sharp vanishing theorems. Furthermore, an additional chapter outlines (without proofs) how the main results on the cohomology of discrete cocompact subgroups extend to general S-arithmetic subgroups of semisimple groups over number fields. This edition can be used as a reference for research mathematicians and advanced graduate students in such diverse fields as representation theory, arithmetic groups, automorphic forms, and algebraic number theory.

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Contents

Introduction to the First Edition

1. This monograph is mainly concerned with two types of cohomology spaces pertaining to a reductive Lie group G (real, p-adic, or product of such groups) and a discrete cocompact subgroup Γ of G. The first one is the Eilenberg-MacLane cohomology space $H^*(\Gamma; E)$ of Γ with coefficients in a finite dimensional unitary Γ-module (or a finite dimensional G-module if G is real). The second one is attached to G, or its Lie algebra $\mathfrak{g}$ and a maximal compact subgroup K if G is real, and a representation V of G, usually infinite dimensional, and appears in various guises: continuous, smooth, or also (for G real) relative Lie algebra cohomology. Our initial interest was in the former one. However, its study may be reduced in part to the latter one (see Chapters VII and XIII), where G is the ambient group and V runs through the irreducible subspaces of $L^2(\Gamma\backslash G)$. The determination of this cohomology is then a first step towards the determination of $H^*(\Gamma: E)$. But, as this work developed, we were led to emphasize it more and more, and to treat it as our main topic rather than as an auxiliary one. In fact, ten out of thirteen chapters are devoted to it, or directly motivated by it.

The material presented here divides naturally into two parts, one devoted mainly to real Lie groups (Chapters I to IX), the other to locally compact totally disconnected groups (for short, t.d. groups), in particular reductive p-adic groups, or products of real Lie groups and t.d. groups (Chapters X to XIII). Each part in turn contains roughly three main items: general results on the cohomology used, specific ones for cohomology and representations of reductive groups, and applications to discrete cocompact subgroups.

We now give some indications on the contents of the various chapters.

2. In Chapters I to VIII, G is a real Lie group with finitely many connected components, and the underlying cohomology is the relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{k}; V)$ or rather, to allow for non-connected G's, a slight modification of it denoted $H^*(\mathfrak{g}, K; V)$. Chapter I is devoted to foundational material on that cohomology. In §§1 to 4, $\mathfrak{g}$ is a finite dimensional Lie algebra over a field of characteristic zero and $\mathfrak{k}$ a subalgebra. §1 recalls the direct definition of $H^*(\mathfrak{g}, \mathfrak{k}; V)$, §2 discusses more generally the derived functors of $\mathrm{Hom}_{\mathfrak{g}}$ in the category $\mathcal{C}_{\mathfrak{g},\mathfrak{k}}$ of $(\mathfrak{g}, \mathfrak{k})$-modules, i.e., $\mathfrak{g}$-modules which are locally finite and semi-simple with respect to $\mathfrak{k}$. This approach differs only in minor details from that of G. Hochschild, in the framework of relative homological algebra. The translation in the formalism of Yoneda's long extensions is briefly recalled in §3. In §4, we give two proofs of a useful vanishing theorem of D. Wigner. From §5 on, $F = \mathbf{R}$, $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ that of a maximal compact subgroup K of G. In §5, we transpose the previous considerations to the category of $(\mathfrak{g}, K)$-modules. In §6 we introduce a slightly different category $\mathcal{C}_{\mathfrak{g},\mathfrak{k},L}$, solely as a tool to prove the existence of a Hochschild-Serre spectral sequence for $(\mathfrak{g}, K)$-modules. Also included are two results of Casselman (5.5)

and of D. Vogan (2.8) on finitely generated or admissible modules, and a Poincaré duality theorem of D. Vogan when G is semi-simple and V irreducible admissible (§7).

Chapter II is devoted to the case where $\mathfrak{g}$ is semi-simple (or reductive) and the coefficient module is the tensor product of a finite dimensional G-module E by a unitary G-module V. The cochain complex for relative Lie algebra cohomology admits then a natural scalar product. Various constructions and results of Matsushima, Matsushima-Murakami, Kuga, originating in differential geometry and Hodge theory and discussed by them in the context of discrete cocompact subgroups, are adapted to our setting in §§1 to 4, and §8; in a similar vein, §§6, 7 prove some vanishing theorems by use of spinors, suggested by results of Hotta and Parthasarathy on discrete subgroups. In §5, we consider the case where V belongs to the discrete series and show, using the characterization of the minimal K-type in V, that $H^q(\mathfrak{g}, K; E \otimes V)$ vanishes unless $2q = \dim G/K$ and V has the same infinitesimal character as the contragredient representation E^* to E.

The main topic of Chapter III is the cohomology with respect to a principal series representation. The computation uses an analogue of Shapiro's lemma (2.5), a description of K-finite vectors in induced representations (2.4), results of B. Kostant on the cohomology of nilpotent radicals of parabolic subalgebras and the Hochschild-Serre spectral sequence (§3). The results are applied in §4 to the determination of the cohomology with respect to tempered representations: in particular, it can be non-zero only in a small interval around the middle dimension and if the underlying parabolic subgroup is fundamental. These results have also been proved independently by G. Zuckerman, and those of §3 for complex semi-simple Lie algebras by P. Delorme. The last paragraph of III contains some general remarks on C^∞-vectors of induced representations, proving in particular that these are smooth functions in the cases of interest to us.

The next step is the investigation of the cohomology with respect to non-tempered representations. It is based on the Langlands classification of irreducible admissible $(\mathfrak{g}, K)$-modules and on two complements to it: some information on the Langlands parameters of the constituents of the kernel of the intertwining operators used by Langlands, and a necessary condition for unitarizability (in fact, for uniform boundedness) in terms of the Langlands parameters. The latter sharpens a result of R. Howe stating that the coefficients of a unitary representation with compact kernel vanish at infinity. These results are proved in Chapter IV (see 4.13, 5.2), which also contains a proof of the Langlands classification (4.11).

The uniform boundedness condition singles out a subset denoted $\Pi_\infty(G)$ of the set $\Pi(G)$ of infinitesimal equivalence classes of irreducible admissible $(\mathfrak{g}, K)$-modules (V, §2). It contains the unitary representations with compact kernel. Chapters V and VI are devoted to the cohomology with coefficients in $\Pi_\infty(G)$, or also in $V \otimes E$, where V represents an element of $\Pi_\infty(G)$ and E is finite dimensional, irreducible. We prove first that $H^q(\mathfrak{g}, K; V \otimes E)$ vanishes for $q < \mathrm{rk}_{\mathbf{R}}\, G$ (3.3), a result also obtained independently by G. Zuckerman. For E trivial, this bound is sharp in $\Pi_\infty(G)$ (but not always in the unitary dual $\widehat{G}$ of G, see (II, 8.7)): in §4, it is shown that the constituents of (an analogue of) the Steinberg representation are all in $\Pi_\infty(G)$, and that $H^q(\mathfrak{g}, K; V) \neq 0$ if $q = \mathrm{rk}_{\mathbf{R}}\, G$ for at least one of them. §5 reproves some results of P. Delorme on the relation between H^1 and the topology of $\widehat{G}$.

Chapter VI gives some further information on the cohomology with respect to a Langlands quotient $J_{P,\sigma,\nu}$. We need only consider the $J_{P,\sigma,\nu}$ with the same infinitesimal character as the trivial representation. The criterion IV, 5.2 gives an upper bound for ν. The general pattern which emerges is that, roughly, the bigger ν (in a suitable order relation), the lower the first non-vanishing cohomology group. Since the cohomology with respect to tempered representations is non-zero only close to the middle dimension, this suggests proceeding by increasing induction on ν. Without attempting to do this in general, we illustrate this relationship in Chapter VI by some general results when ν is minimal (§§1, 2) or $\mathrm{rk}_{\mathbf{R}}\, G = 1$ (§3), and by a complete determination of the cohomology when $G = \mathbf{SO}(n,1)$, $\mathbf{SU}(n,1)$ in §4.

Chapter VII is devoted to the cohomology of discrete subgroups. First if Γ is a discrete subgroup of the Lie group G, and E is a G-module, then we have the (well-known) formula

$$H^*(\Gamma; E) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E) \tag{1}$$

(2.7). If now Γ is cocompact, then $L^2(\Gamma\backslash G)$ admits a Hilbert discrete sum decomposition with finite multiplicities

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} m(\pi,\Gamma)H_\pi, \tag{2}$$

and (1) transforms to

$$H^*(\Gamma; E) = \bigoplus_{\pi\in\widehat{G}} m(\pi,\Gamma)H^*(\mathfrak{g}, K; H_\pi \otimes E) \tag{3}$$

(5.2). There is also a counterpart to that formula when E is a unitary Γ-module, involving the decomposition of the unitarily induced representation $I^G_{\Gamma,2}(E)$ (3.2). Various consequences of the results of the previous chapters are drawn in §§4, 6.

Chapter VIII is concerned with cohomology at the $\mathbf{R}$-rank q when $G = \mathbf{SU}(p,q)$ $(p \geq q)$. Let F_ℓ be the irreducible G-module whose highest weight is ℓ times the highest weight of the standard representation of $\mathbf{SU}(p,q)$ in $\mathbf{C}^{p+q}$. For each $\ell \geq q$ there is a unitary irreducible representation H_ℓ of G such that $H^q(\mathfrak{g}, K; H_\ell \otimes F_{\ell-q}) \neq 0$ (2.13). It is then shown that certain cocompact arithmetically defined subgroups of G have subgroups of finite index Γ' such that H_ℓ occurs in $L^2(\Gamma\backslash G)$, whence in particular $H^q(\Gamma'; F_{\ell-q}) \neq 0$. This extends a result of Kazhdan concerning the case where $q = 1$, which gave the first examples of discrete cocompact subgroups of $\mathbf{SU}(n,1)$ with non-vanishing first Betti number for arbitrary n. The proof uses the metaplectic representation and the duality theorem, and is quite similar to that of Kazhdan, although the context is a bit different, since Kazhdan worked with adelic groups.

3. Chapters IX to XII are devoted to continuous and smooth cohomology. §§1 to 4 of Chapter IX contain some basic material concerning derived functors in the category $\mathcal{C}_G$ of continuous G-modules (always assumed to be locally convex Hausdorff topological vector spaces over $\mathbf{C}$), when G is a locally compact group (countable at infinity). The approach is the one of Hochschild-Mostow, based on

the use of injective modules relative to G-morphisms which are strong (i.e. split for the underlying structure of topological vector spaces). After that, we are concerned with real groups (IX, §§5, 6), t.d. (totally disconnected) groups, in particular p-adic groups (X, XI), and products of such groups (XII). The formal analogies between these three cases are emphasized. In each, besides $\mathcal{C}_G$, we consider the categories $\mathcal{C}_G^\infty$ of smooth topological G-modules and $\mathcal{C}_G^f$ of non-degenerate modules over a suitable Hecke algebra. The last one (introduced in substance by Jacquet-Langlands) is abelian and the modules in it are just complex vector spaces. The Hecke algebras occurring here have no unit in general, a situation not considered in standard texts on homological algebra. However they are idempotented, and this allows one to extend some standard constructions to our case (XII, §0). In particular, $\mathcal{C}_G^f$ has enough injectives. There are natural functors

$$\mathcal{C}_G \underset{\alpha}{\overset{\gamma}{\rightleftarrows}} \mathcal{C}_G^\infty \xrightarrow[\beta]{} \mathcal{C}_G^f,$$

where α (resp. β) is the passage to smooth (resp. K-finite vectors) and γ is the inclusion. γ preserves derived functors and β cohomology for quasi-complete spaces, α preserves derived functors for quasi-complete spaces in the t.d. case, and cohomology for Fréchet spaces in the other two cases.

In the real case, $\mathcal{C}_G^\infty$ consists of the usual differentiable modules, with the C^∞-topology, while, up to Chapter IX, $\mathcal{C}_G^f$ is just the category of $(\mathfrak{g}, K)$-modules. But, as is known, it may also be viewed as the category of non-degenerate modules over the Hecke algebras $\mathcal{H}(\mathfrak{g}, K)$ of bi-K-finite distributions on G with support in K. This point of view is more convenient to treat the mixed case, and is introduced later (XII, §2). The above conservation theorems for derived functors in the real case (due to Hochschild-Mostow, W. v. Est, P. Blanc) are proved in IX, §§5, 6.

If G is a t.d. group (X, §1), then a topological G-module V is smooth if every $v \in V$ is fixed under an open subgroup and V is, topologically, the inductive limit of the subspaces V^{L^*} of fixed points under compact open subgroups L of G. The Hecke algebra underlying the definition of $\mathcal{C}_G^f$ is the convolution algebra of locally constant compactly supported functions. The main case of interest is when $G = \mathcal{G}(k)$, where k is a non-archimedean local field and $\mathcal{G}$ a connected reductive k-group. If $V \in \mathcal{C}_G$, then the V-valued cochains of the Bruhat-Tits building of G provide an s-injective resolution of V (X, §2). In §4 of X we prove the results of W. Casselman which give a complete description of the cohomology of G with respect to an irreducible admissible G-module. §5 is devoted to $\mathcal{C}_G^f$, and the passage to $\mathcal{C}_G^f$ is used in §6 to prove some Künneth rules.

Chapter XI is a p-adic counterpart of IV. It discusses the analogue of the Langlands classification, and of the uniform boundedness condition. The latter is used to show that the only irreducible admissible representations with compact kernel, with respect to which G has non-vanishing cohomology in some dimension $q \neq 0$, $\mathrm{rk}_k\, \mathcal{G}$, are non-unitarizable (a result due to W. Casselman).

Let now $G = G_1 \times G_2$ be the product of a real Lie group G_1 and a t.d. group G_2. A topological continuous G-module V is said to be smooth if it is smooth with respect to G_1 and G_2 and if it is the topological inductive limit of the subspaces V^L, where L runs through the compact open subgroups of G_2.

There are also intermediate categories of continuous G-modules smooth with respect to one of the factors. The relations between the corresponding derived functors are discussed in §1. In §2, we fix a maximal compact subgroup K_1 of G_1 and pass to the $(K_1 \times L)$-finite vectors, where L is a compact open subgroup of G_2, which brings us to the non-degenerate modules over the Hecke algebra $\mathcal{H}(G) = \mathcal{H}(\mathfrak{g}_1, K_1) \otimes \mathcal{H}(G_2)$. §3 is devoted to some Künneth rules and to applications to the cohomology of products of reductive groups or of adelic groups.

In Chapter XIII, we consider the cohomology space $H^*(\Gamma; E)$, where Γ is a discrete cocompact subgroup of G and E a finite dimensional unitary Γ-module, first in general (§1), then when G is a product of reductive groups G_s $(s \in S)$. In the latter case, we have a formula quite similar to (3), except that $L^2(\Gamma\backslash G)$ is replaced by the unitarily induced representation from E. Furthermore, since the G_s's are of type I, each $\pi \in \widehat{G}$ is a Hilbert tensor product $\pi = \widehat{\bigotimes}_s \pi_s$ $(\pi_s \in \widehat{G}_s)$, and the Künneth rule gives

$$H^*_{ct}(G; H_\pi) = \bigotimes_s H_{ct}(G_s; H_{\pi_s}). \tag{4}$$

This allows us to apply the earlier results on continuous cohomology of real or p-adic groups. We then pass to some applications. We prove the Casselman vanishing theorem (2.6) and extend it to the case where Γ is irreducible (3.1) in a product of semi-simple groups over non-archimedean fields (3.6). Following a suggestion of G. Prasad, we also show it to be valid when E is a finite dimensional vector space over an arbitrary field of characteristic zero, and G has rank ≥ 2, using a theorem of Margulis (3.7). Finally, we prove that if $G = \mathcal{G}(A)$ is the adèle group of a semi-simple anisotropic group $\mathcal{G}$ over a global field, then $H^*(\mathcal{G}(k); \mathbf{R})$ reduces to the continuous cohomology of the archimedean factor of $\mathcal{G}(A)$ (3.9).

A survey of some of the main results on vanishing and non-vanishing cohomology is given at the end of the book.

4. This monograph is an outgrowth of a seminar on the "Cohomology of discrete subgroups of semi-simple Lie groups" held at The Institute for Advanced Study in 1976–77. A first set of notes was written and distributed at that time. Most of the material of these notes is incorporated in Chapters I to IX, except for some results which were rendered somewhat obsolete by others found in the course of the seminar. There was also some discussion of the p-adic case in the seminar, but it was not written up then. In the first version, we kept track of who did what and each chapter was accordingly authored or coauthored. It would have been quite awkward to do so in the present version, which represents a considerable reorganization and expansion of the first one. Rather, we prefer to take joint responsibility for the results and mistakes in this book, except however that the first (resp. second) named author wishes to leave credit for Chapters IV, VIII, XI (resp. VII, IX, XII, XIII) to the second (resp. first) named author.

The transition from the first to the final version was a rather painful process, involving a long series of changes, additions, amplifications, corrections upon corrections, reshuffling and renumbering. We are very grateful to the secretaries of the School of Mathematics, and in particular to Peggy Murray, who had by far the greatest load, for having taken care so skillfully and so speedily of this endless series of changes upon changes, which required expertise not only in typing but in cutting, pasting and collage as well.

A reference such as 3.4 (resp. 3.4(1)) refers to section 3.4 (resp. relation 3.4(1)) of the same chapter; if preceeded by a capitalized Roman numeral it refers to the corresponding section or relation of the chapter denoted by that numeral.

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Introduction to the Second Edition

This second edition includes a number of corrections, minor changes or amplifications to the original text, as well as some further material that reports on later relevant developments.

The numbering in the first edition has been maintained. The new additions have been inserted either at the beginning or the end of a paragraph, or a chapter. This explains some numbering that is a bit unusual: In section 3 of Chapter 0, in particular, there is a subsection 3.0 (which has subsections). The main new topics are:

I, §8, which gives a construction, in the framework of this book, of the Zuckerman functors and describes their main properties.

II, §10 provides sharp bounds, case by case, for the vanishing theorems, due to Enright, Kumaresan, Parthasarathy, Vogan-Zuckerman, which in many cases are improvements of the ones given originally.

VI, §0 introduces the translation functors and their relationship with relative Lie algebra cohomology.

VI, §5 is devoted to the Vogan-Zuckerman theorem, which describes $\mathrm{Ext}^*_{\mathfrak{g},K}(F, V)$, where V runs through the irreducible unitary $(\mathfrak{g}, K)$-modules and F through the finite dimensional irreducible $(\mathfrak{g}, K)$-modules.

XIII, §4 studies the cohomology of an S-arithmetic subgroup of G with coefficients in a rational G-module.

Moreover, a new Chapter XIV has been added. It outlines how the main results proved in Chapters VII, VIII and XIII for the cohomology of discrete cocompact subgroups extend to general S-arithmetic subgroups of semisimple algebraic groups over number fields.

It has been almost 20 years since the publication of the original version of this book. During that time the methods of homological algebra have become increasingly important in the construction of admissible representations and in the study of arithmetic groups. Although some of the original material in this book has been superseded, it is still a useful reference. We thank the American Mathematical Society, in particular S. Gelfand, for having encouraged us to publish this second edition. The authors would also like to thank the editorial staff for an extremely helpful and thorough reading of the manuscript.

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CHAPTER 0

Notation and Preliminaries

§1 contains some general notation, §2 some definitions and facts on representations of Lie groups, and §3 fixes a number of conventions on reductive groups. The notation introduced here will often be used without reference.

1. Notation

1.1. As usual, $\mathbf{Z}$ is the ring of integers, $\mathbf{N} = \{z \in \mathbf{Z} \mid z \geq 0\}$ the set of natural integers, $\mathbf{Q}$ (resp. $\mathbf{R}$, resp. $\mathbf{C}$) the field of rational (resp. real, resp. complex) numbers, $\mathbf{R}_+^*$ the multiplicative group of strictly positive real numbers.

If A is an algebra with identity, then A^* is the group of units of A.

1.1.1. If $V = \bigoplus_{i \in \mathbf{Z}} V^i$ is a vector space graded by $\mathbf{Z}$ and if $m \in \mathbf{Z}$, then $V[m]$ denotes the graded vector space defined by

$$V[m]^i = V^{i+m} \ (i \in \mathbf{Z}).$$

1.1.2. Let V be a complex vector space. If V has the structure of a module over a group or a Lie algebra and if $m \in \mathbf{N}$, then we have consistently written mV for the direct sum of m copies of V, with the corresponding diagonal action, thus committing an abuse of notation. To adopt a correct one would entail an amount of changes that we found too daunting. We thereby, regretfully, announce that we shall maintain our original convention.

1.2. If G is a group, and M a subset of G, then $\mathcal{Z}_G(M)$ or $\mathcal{Z}(M)$ is the centralizer of M and $\mathcal{N}_G(M)$ or $\mathcal{N}(M)$ the normalizer of M:

$$\mathcal{Z}_G(M) = \{g \in G \mid g \cdot m = m \cdot g \ (m \in M)\},$$
$$\mathcal{N}_G(M) = \{g \in G \mid g \cdot M \cdot g^{-1} \subset M\}.$$

$\operatorname{Int} g$ is the inner automorphism $x \mapsto g \cdot x \cdot g^{-1}$. We also write ${}^g x$ for $\operatorname{Int} g(x)$, and ${}^g M = \operatorname{Int} g(M)$. The center of G is denoted $\mathcal{Z}(G)$ or $\mathcal{C}(G)$, and $\mathcal{D}G$ is the derived group of G.

1.3. If $g \in G$, then ℓ_g (resp. r_g) denotes the left (resp. right) translation by g on G, or on functions f on G. In particular

$$(1) \qquad \ell_g f(x) = f(g^{-1} \cdot x), \qquad r_g f(x) = f(x \cdot g) \qquad (x \in G).$$

Thus $\ell_{g \cdot h} = \ell_g \cdot \ell_h$, $r_{gh} = r_g \cdot r_h$ $(g, h \in G)$.

1.4. If G is a topological group, then G^0 is the connected component of the identity in G.

1.5. The Lie algebra of a real Lie group $G, H, \cdots$ will be denoted by the corresponding German lower case letter $\mathfrak{g}, \mathfrak{h}, \cdots$, and the exponential map $\mathfrak{g} \to G$ is denoted exp. We also write e^x for $\exp x (x \in \mathfrak{g})$. If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, then $\mathfrak{m}_c$ stands for the complexification $m \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{m}$.

The universal enveloping algebra over $\mathbf{C}$ of $\mathfrak{g}$ is denoted $U(\mathfrak{g})$. Its center is denoted $Z(\mathfrak{g})$.

The centralizer (resp. normalizer) of $\mathfrak{m}$ in $\mathfrak{g}$ is denoted $\mathfrak{z}(\mathfrak{m})$ or $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{m})$, resp. $\mathfrak{n}(\mathfrak{m})$ or $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})$:

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{m}) = \{x \in \mathfrak{g}/[\mathfrak{m}, x] = 0\}, \qquad \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}) = \{x \in \mathfrak{g}/[x, m] \in \mathfrak{m}\ (m \in \mathfrak{M})\}.$$

As usual the differential of $\operatorname{Int} x$ $(x \in G)$ at 1 is denoted $\operatorname{Ad} x$, and, for $x \in \mathfrak{g}$, $\operatorname{ad} x \colon \mathfrak{g} \mapsto \mathfrak{g}$ is defined by $\operatorname{ad} x(y) = [x, y]$. For $\mathfrak{m} \subset \mathfrak{g}$, we let

$$\mathcal{Z}_G(\mathfrak{m}) = \{x \in G \mid \operatorname{Ad} x(m) = m \ (m \in \mathfrak{m})\},$$
$$\mathcal{N}_G(\mathfrak{m}) = \{x \in G \mid \operatorname{Ad} x(\mathfrak{m}) = \mathfrak{m}\}.$$

1.6. If G is a Lie group, then $X(G)$ is the group of continuous homomorphisms of G into $\mathbf{R}^*$ and

$$^0G = \bigcap_{\chi \in X(G)} \ker |\chi|.$$

It is a normal subgroup which contains the derived group and all compact subgroups of G.

1.7. Unless otherwise stated, topological vector spaces are assumed to be over $\mathbf{R}$ or $\mathbf{C}$, Hausdorff locally convex and quasi-complete, and manifolds to be C^∞ and countable at infinity. If M is a manifold and V a topological vector space, then $C^\infty(M; V)$ is the space of C^∞-functions of M, with values in V, endowed with the C^∞-topology. The space of V-valued smooth differential p-forms ($p \in \mathbf{N}$) on M is denoted $A^p(M; V)$, and $A^*(M; V)$ is the direct sum of the spaces $A^p(M; V)$. Thus $A^0(M; V) = C^\infty(M; V)$. If V is a Fréchet space, then so is $A^p(M; V)$ $(p \in \mathbf{N})$.

If M, N are manifolds, then $C^\infty(A, B)$ is the space of smooth maps $A \to B$, endowed with the C^∞-topology.

2. Representations of Lie groups

2.1. Let G be a Lie group with finite component group. By a topological G-module (or simply a G-module) V, where V is assumed to be a locally convex and locally complete Hausdorff topological vector space over $\mathbf{C}$, we mean a homomorphism $G \to \operatorname{Aut} V$ defined by a *continuous* map $G \times V \to V$. It will be denoted (π, V), or V or π. The action of g on v is often denoted $g.v$ or gv rather than $\pi(g)v$. We shall denote by $\mathcal{C}_G$ the category of topological G-modules and equivariant continuous linear maps.

V is said to be finitely generated if there is a finite subset S of V such that the span of the vectors $g.c$ $(g \in G, c \in S)$ is dense in V.

2.2. Let $(\pi, V) \in \mathcal{C}_G$. For $v \in V$ we let $c_v \colon G \to V$ denote the orbit map $c_v(g) = \pi(g)v$. It is continuous. If $\widetilde{v}$ is a continuous functional on V, then the function $c_{v,\widetilde{v}}$ on G defined by

$$c_{v,\widetilde{v}}(g) = \langle \pi(g)v, \widetilde{v} \rangle = \langle c_v(g), \widetilde{v} \rangle \qquad (g \in G) \tag{1}$$

is called a *coefficient* of π.

An elementary calculation shows that we have

$$c_{x\cdot v,\tilde{v}} = r_x c_{v,\tilde{v}} \qquad (x \in G). \tag{2}$$

If V is a Hilbert space, then the coefficients may also be defined to be the functions $c_{v,w}\colon g \mapsto (\pi(g)v, w)$, where $v, w \in V$ and $(\ ,\ )$ is the scalar product on V.

2.3. Let $(\pi, V) \in \mathcal{C}_G$. The vector $v \in V$ is said to be differentiable (resp. analytic) if c_v is C^∞ (resp. analytic). The space of differentiable (resp. analytic) vectors is denoted V^∞ (resp. V^ω). It is stable under G. The representation π defines a representation of $\mathfrak{g}$ or $U(\mathfrak{g})$ on V^∞ (resp. V^ω) which is denoted π_∞ (resp. π_ω) or simply π.

A continuous representation (π, V) is differentiable if $V = V^\infty$ and if the map $v \mapsto \{g \mapsto g \cdot v\}$ is a topological isomorphism of V onto its image in $C^\infty(G; V)$, endowed with the topology induced from that of $C^\infty(G; V)$, to be called the C^∞-topology. We let $\mathcal{C}_G^\infty$ be the category of differentiable G-modules and continuous G-morphisms.

Let (π, V) be a continuous G-module. Then V^∞, endowed with the C^∞-topology and the given action of G, is a differentiable G-module. We denote it (π_∞, V^∞). If V is a Fréchet space, then so is V^∞. The map $(\pi, V) \mapsto (\pi_\infty, V^\infty)$ is a functor. If V is a Hilbert space, then, by the principle of uniform boundedness, the topology on V^∞ is defined by the semi-norms $v \mapsto \|Xv\|$ $(X \in U(\mathfrak{g}))$.

2.4. A vector $v \in V$ is *G-finite* if it is contained in a finite dimensional subspace stable under G. A G-module is *locally finite* if every element is G-finite.

Let K be a compact subgroup of G. We let $V_{(K)}$ denote the space of K-finite vectors. It is the union of the images $V_{(W)}$ of the maps

$$\mathrm{Hom}_K(W, V) \otimes W \to V$$

defined by

$$\tau \otimes w \mapsto \tau(w) \qquad (\tau \in \mathrm{Hom}_K(W, V), w \in W),$$

where W runs through all finite dimensional K-modules. If W is irreducible, then $V_{(W)}$ is the *isotypic subspace* of type W. We say that V is *admissible* if all isotypic subspaces are finite dimensional (or equivalently all of the $V_{(W)}$ are finite dimensional for all finite dimensional W).

Assume a maximal compact subgroup K of G has been fixed. Then we set $V_0 = V^\infty \cap V_{(K)}$. This space is stable under $\mathfrak{g}$. Note that if an isotypic subspace of K in V is finite dimensional, then it is contained in V^∞. We say that π is *admissible* if the isotypic subspaces in V are all finite dimensional. In this case $V_0 = V_{(K)}$.

2.5. A $(\mathfrak{g}, K)$-module is a real or complex vector space which is a $\mathfrak{g}$-module, a locally finite and semi-simple K-module and such that the operations of $\mathfrak{g}$ and K satisfy the following compatibility conditions:

1) $\pi(k) \cdot (\pi(X)) \cdot v = \pi(\mathrm{Ad}\, k(X)) \cdot \pi(k) \cdot v$ $(k \in K; X \in U(\mathfrak{g}); v \in V)$;

2) if F is a K-stable finite dimensional subspace of V, then the representation of K on F is differentiable, and has $\pi|_{\mathfrak{k}}$ as its differential.

A $(\mathfrak{g}, K)$-module is *admissible* if it is admissible as K-module.

Let V be a vector space on which $\mathfrak{g}$ and K operate so as to satisfy 1) and 2) and in which every K-stable finite dimensional subspace is K-semi-simple. Then the subspace $V_{(K)}$ of K-finite vectors in V is K-semi-simple and stable under $\mathfrak{g}$, hence is a $(\mathfrak{g}, K)$-module.

If (π, V) is a $(\mathfrak{g}, K)$-module, then $\mathfrak{g}$ and K operate as usual on the dual space V' of V. The above conditions are met. The space of K-finite vectors in V' is then a $(\mathfrak{g}, K)$-module, to be called the *contragredient* (g, K)*-module to* V, and to be denoted $(\widetilde{\pi}, \widetilde{V})$. It is admissible if and only if V is. In that case, V is contragredient to $\widetilde{V}$.

A $(\mathfrak{g}, V)$-module (π, V) is *unitary* if V is endowed with a positive non-degenerate scalar product (,) which is invariant under K and (infinitesimally) invariant under $\mathfrak{g}$:

$$(\pi(k) \cdot v, \pi(k) \cdot w) = (v, w),$$
$$(\pi(x)v, w) + (v, \pi(x) \cdot w) = 0 \quad (v, w \in V, k \in K, x \in \mathfrak{g}).$$

We let $\mathcal{C}_{\mathfrak{g},K}$ be the category of $(\mathfrak{g}, K)$-modules and $(\mathfrak{g}, K)$-morphisms, and $\Pi(G)$ the set of isomorphisms classes of irreducible admissible $(\mathfrak{g}, K)$-modules.

A $(\mathfrak{g}, K)$-module (π, V) (or a differentiable G-module) is said to have an *infinitesimal character* χ if there is a homomorphism $Z(\mathfrak{g}) \to \mathbf{C}$ such that $\pi(z) = \chi(z) \cdot \mathrm{Id}$ for all $z \in Z(\mathfrak{g})$. This is in particular the case if (π, V) is irreducible and admissible.

2.6. Let $(\pi, V) \in \mathcal{C}_G$. Then V_0 is a $(\mathfrak{g}, K)$-module. We denote it sometimes (π_0, V_0). It is admissible (resp. unitary) if (π, V) is so, and it is finitely generated as a $\mathfrak{g}$-module if (π, V) is finitely generated as a G-module.

It is known that every irreducible admissible $(\mathfrak{g}, K)$-module can be realized as the space of K-finite vectors in an irreducible admissible differentiable G-module [**77**]. In fact, this statement is true more generally for finitely generated admissible $(\mathfrak{g}, K)$-modules, but we shall not need this fact.

Two smooth representations are *infinitesimally equivalent* if the two associated $(\mathfrak{g}, K)$-modules of K-finite vectors are isomorphic.

2.7. We let $Z(\mathfrak{g}, K)$ denote the subgroup of elements of the center of K which act trivially on $\mathfrak{g}$. If G is connected, with compact center, then $Z(\mathfrak{g}, K)$ is just the center of G. We say that a $(\mathfrak{g}, K)$-module (π, V) has a central character ω_π if there exists a character $\omega_\pi \colon Z(\mathfrak{g}, K) \to \mathbf{C}^*$ such that $\pi(z) = \omega_\pi(z) \cdot \mathrm{Id}$ for all $z \in Z(\mathfrak{g}, K)$. If (π, V) is admissible and irreducible, then it has both an infinitesimal character and a central character.

2.8. The set of equivalence classes of irreducible unitary representations of G is denoted $\mathcal{E}(G)$ or $\widehat{G}$.

Let (π, V) be unitary, irreducible. There exists then a unitary character ω_π of $\mathcal{C}(G)$ such that $\pi(z) = \omega_\pi(z)\,\mathrm{Id}$ for $z \in \mathcal{C}(G)$. Therefore $|c_{u,v}|$ $(u, v \in V)$ is a function on $G/\mathcal{C}(G)$. The representation π is said to be in the discrete series if it is unitary, irreducible and if its coefficients are square integrable modulo the center, i.e. on $G/\mathcal{C}(G)$. We let $\mathcal{E}_d(G)$ be the set of equivalence classes of discrete series representations of G.

If G is compact, then $\mathcal{E}(G) = \mathcal{E}_d(G)$.

3. Linear algebraic and reductive groups

3.0. In this book, up to Chapter XII, we are mainly concerned with real or complex Lie groups. The point of view of algebraic groups becomes more prominent in XIII, XIV. Our general reference for linear algebraic groups is [**124**]. We review some basic concepts in characteristic 0.

k is a field of characteristic 0, and K an algebraically closed extension of k.

3.0.1. A subgroup $\mathcal{G} \subset \mathbf{GL}_n(K)$ is *linear algebraic* if there exist polynomials $P_\alpha \in K[X_{11}, X_{12}, \ldots, X_{nn}]$, $\alpha \in I$, such that

$$\mathcal{G} = \{g = (g_{ij}) \in \mathbf{GL}_n(K) \mid P_\alpha(g_{11}, \ldots, g_{nn}) = 0 \ (\alpha \in I)\}.$$

It is defined over k if the ideal of polynomials vanishing on $\mathcal{G}$ is generated by elements of $k[X_{11}, X_{12}, \ldots, X_{nn}]$. Then we set $\mathcal{G}(k) = \mathcal{G} \cap \mathbf{GL}_n(k)$.

The group $\mathcal{G}$ is connected (in the Zariski topology) if and only if it is irreducible as an algebraic variety. If $K = \mathbf{C}$, $\mathcal{G}$ is also a complex Lie group, and it is connected if and only if it is connected as a manifold. Moreover, if it is defined over $\mathbf{R}$, then $\mathcal{G}(\mathbf{R})$ is a Lie group which may have several (but at most finitely many) connected components in the ordinary topology.

3.0.2. The group $\mathbf{GL}_1$ may be identified with the group K^*. The linear algebraic group $\mathcal{G}$ is an (algebraic) *torus* if it is isomorphic to a product of a finite number of $\mathbf{GL}_1$'s. This is equivalent with the requirement that it is diagonalizable. If $\mathcal{G}$ is moreover defined over k and the isomorphism can be defined over k, then it is said to be split over k or *k-split*. (This condition is equivalent with the requirement that there exist $g \in \mathbf{GL}_n(k)$ such that $g\mathcal{G}g^{-1}$ is diagonal.)

3.0.3. Let $\mathcal{G}$ be a linear algebraic group defined over k. The maximal k-split tori of $\mathcal{G}$ are all conjugate under $\mathcal{G}(k)$. Their common dimension is the *k-rank*, $\mathrm{rk}_k(\mathcal{G})$ of $\mathcal{G}$ [**18**] (see also [**124**], 2.0.9, 19.2).

3.0.4. The group $\mathcal{G}$ is reductive if its Lie algebra is reductive.

Assume that $\mathcal{G}$ is connected. Then a closed subgroup $\mathcal{P}$ of $\mathcal{G}$ is parabolic if $\mathcal{G}/\mathcal{P}$ is a projective variety.

3.1. In this book, a real Lie group G is said to be *reductive* if there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbf{R}$, whose identity component (in the Zariski topology) is reductive and a morphism $\nu\colon G \to \mathcal{G}(\mathbf{R})$ with finite kernel, whose image is an open subgroup of finite index of $\mathcal{G}(\mathbf{R})$. Unless otherwise stated, we also assume that G is of "connected type", i.e. that $\mathrm{Ad}\, G$ is contained in $\mathrm{Ad}(\mathfrak{g}_c)$.

This implies in particular that the identity component $\mathcal{Z}(G^0)^0$ of the center of G^0 is also central in G.

3.2. The usual terminology of algebraic groups will be extended to such groups. In particular, a subgroup T of G is a torus (resp. $\mathbf{R}$-split torus) if it is the inverse image of the group of real points $\mathcal{S}(\mathbf{R})$ of an $\mathbf{R}$-torus (resp. $\mathbf{R}$-split torus) $\mathcal{S}$ of $\mathbf{G}$. The *split component of a torus* T is the identity component of its greatest $\mathbf{R}$-split subtorus. The maximal $\mathbf{R}$-split tori of G are conjugate under G^0. Their common dimension is the $\mathbf{R}$-rank or split rank $\mathrm{rk}_{\mathbf{R}}(G)$ of G. The *split component* of G is the identity component of the greatest split torus in the center of G (or, equivalently, of G^0, cf. 3.1). The group G is the direct product of its split component by ${}^0G$.

3.3. A *Cartan involution* θ of G is an involutive automorphism of G whose fixed point set is a maximal compact subgroup and which is the inversion on the split component of G. Given K, there is exactly one Cartan involution with fixed point set K. If $\mathfrak{s}$ is the (-1)-eigenspace of $d\theta$, then $(k, x) \mapsto k \cdot \exp x$ $(k \in K, x \in \mathfrak{s})$ is an isomorphism of analytic manifolds of $K \times \mathfrak{s}$ onto G. In particular $S = \exp \mathfrak{s}$ is a closed subspace isomorphic to $\mathfrak{s}$ under the exponential mapping, on which θ acts by inversion. The Cartan involutions are conjugate under automorphisms of G.

3.4. A *parabolic subgroup* P is the normalizer of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. It is the inverse image of the group of real points $\mathcal{P}(\mathbf{R})$ of a parabolic subgroup $\mathcal{P}$ defined over $\mathbf{R}$ of $\mathcal{G}$. The unipotent radical N or N_P of P is the analytic subgroup generated by the nilradical of $\mathfrak{p}$. A Levi subgroup M of P is the inverse image of a Levi $\mathbf{R}$-subgroup $\mathcal{M}$ of $\mathcal{P}$. A *split component* A of P is the split component of a maximal torus in the radical of P. If A_P or A is one, then it is a split component of $\mathcal{Z}_G(A)$, and $\mathcal{Z}_G(A)$ is a Levi subgroup of P. We have

$$(1)\qquad P = M \ltimes N, \quad M = A \times {}^0M, \quad \text{hence} \quad P = MN = A \cdot {}^0M \cdot N.$$

In particular, $P \cap \theta(P)$ is the unique θ-stable Levi subgroup of P. Its split component is $P \cap S$. We always have $G = P \cdot K$, and $K \cap P$ is a maximal compact subgroup of $P \cap \theta(P)$. The dimension of A is the *parabolic rank* $\mathrm{prk}(P)$ of P.

A p-pair is a pair (P, A) consisting of a parabolic subgroup and a split component A of P. The standard Levi decomposition of P is $P = M \cdot N$ with $M = \mathcal{Z}_G(A)$. A p-pair (P', A') dominates (P, A) (written $(P', A') \succ (P, A)$) if $P' \supset P$, $A' \subset A$. The minimal p-pairs are conjugate under inner automorphisms of G^0, or even K^0. If a minimal parabolic subgroup P_0 (resp. a minimal p-pair (P_0, A_0)) is chosen, the *standard parabolic subgroups* (resp. p-pairs) are the parabolic subgroups containing P_0 (resp. the p-pairs dominating (P_0, A_0)). A p-pair (P, A) is *semi-standard* if $A \subset A_0$.

The p-pair $(\overline{P}, A)$ opposite to (P, A) consists of A and of the parabolic subgroup $\overline{P}$ opposite to P and containing $M = \mathcal{Z}_G(A)$. Thus $\overline{P} = M \cdot \overline{N}$, $\overline{N} = N_{\overline{P}}$. If M is θ-stable, then $\overline{P} = \theta(P)$.

3.5. Let (P, A) be a p-pair. Then $\Phi(P, A)$ is the set of roots of P with respect to A and $\Delta(P, A)$ the set of simple roots in $\Phi(P, A)$. We shall indifferently view it also as the set $\Phi(\mathfrak{p}, \mathfrak{a})$ of roots of $\mathfrak{p}$ with respect to $\mathfrak{a}$, i.e. we make no distinction between a character α of A and its differential. The value of a character α on $a \in A$ is denoted $\alpha(a)$ or a^α. Moreover we let

$$(1)\qquad \rho_P(a) = \left(\det \mathrm{Ad}\, a\big|_{\mathfrak{n}}\right)^{1/2} \quad (a \in A);$$

more generally

$$(2)\qquad \rho_P(m) = \left|\left(\det \mathrm{Ad}\, m\big|_{\mathfrak{n}}\right)\right|^{1/2} \quad (m \in M),$$

where $P = M \cdot N$ is the standard Levi decomposition of P. Thus, in the Lie algebra language, ρ_P is half the sum of the elements of $\Phi(\mathfrak{p}, \mathfrak{a})$, each counted with its multiplicity. Every element of $\Phi(P, A)$ is a linear combination with coefficients in $\mathbf{N}$ of elements in $\Delta(P, A)$. The latter are linearly independent, and their number is equal to $\dim A \cap \mathcal{D}G$. We have $\Phi(P, A) = -\Phi(\overline{P}, A)$.

3.6. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\Phi = \Phi(\mathfrak{g}_c, \mathfrak{h}_c)$ is the set of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$. If $\mathfrak{a}_0$ is the Lie algebra of a maximal $\mathbf{R}$-split torus, then ${}_{\mathbf{R}}\Phi = {}_{\mathbf{R}}\Phi(\mathfrak{g}, \mathfrak{a}_0)$ is the set of $\mathbf{R}$-roots, i.e. of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}_0$. The algebras $\mathfrak{a}_0$ are the Lie algebras of the split components of the minimal parabolic subgroups of G. If (P_0, A_0) is a minimal p-pair, then

$${}_{\mathbf{R}}\Phi(\mathfrak{g}, \mathfrak{a}) = \Phi(P_0, A_0) \cup (-\Phi(P_0, A_0)) = \Phi(P_0, A_0) \cup \Phi(\overline{P}_0, A_0),$$

and $\Phi(P_0, A_0)$ is the set of positive elements in ${}_R\Phi(\mathfrak{g}, \mathfrak{a})$ for some ordering.