Differential Equations,
Dynamical Systems,
and Linear Algebra

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Differential Equations, Dynamical Systems, and Linear Algebra

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Preface

This book is about dynamical aspects of ordinary differential equations and the relations between dynamical systems and certain fields outside pure mathematics. A prominent role is played by the structure theory of linear operators on finite-dimensional vector spaces; we have included a self-contained treatment of that subject.

The background material needed to understand this book is differential calculus of several variables. For example, Serge Lang's *Calculus of Several Variables*, up to the chapter on integration, contains more than is needed to understand much of our text. On the other hand, after Chapter 7 we do use several results from elementary analysis such as theorems on uniform convergence; these are stated but not proved. This mathematics is contained in Lang's *Analysis I*, for instance. Our treatment of linear algebra is systematic and self-contained, although the most elementary parts have the character of a review; in any case, Lang's *Calculus of Several Variables* develops this elementary linear algebra at a leisurely pace.

While this book can be used as early as the sophomore year by students with a strong first year of calculus, it is oriented mainly toward upper division mathematics and science students. It can also be used for a graduate course, especially if the later chapters are emphasized.

It has been said that the subject of ordinary differential equations is a collection of tricks and hints for finding solutions, and that it is important because it can solve problems in physics, engineering, etc. Our view is that the subject can be developed with considerable unity and coherence; we have attempted such a development with this book. The importance of ordinary differential equations *vis à vis* other areas of science lies in its power to motivate, unify, and give force to those areas. Our four chapters on "applications" have been written to do exactly this, and not merely to provide examples. Moreover, an understanding of the ways that differential equations relates to other subjects is a primary source of insight and inspiration for the student and working mathematician alike.

Our goal in this book is to develop nonlinear ordinary differential equations in open subsets of real Cartesian space, $\mathbf{R}^n$, in such a way that the extension to manifolds is simple and natural. We treat chiefly autonomous systems, emphasizing qualitative behavior of solution curves. The related themes of stability and physical significance pervade much of the material. Many topics have been omitted, such as Laplace transforms, series solutions, Sturm theory, and special functions.

The level of rigor is high, and almost everything is proved. More important, however, is that *ad hoc* methods have been rejected. We have tried to develop proofs that add insight to the theorems and that are important methods in their own right.

We have avoided the introduction of manifolds in order to make the book more widely readable; but the main ideas can easily be transferred to dynamical systems on manifolds.

The first six chapters, especially Chapters 3–6, give a rather intensive and complete study of linear differential equations with constant coefficients. This subject matter can almost be identified with linear algebra; hence those chapters constitute a short course in linear algebra as well. The algebraic emphasis is on eigenvectors and how to find them. We go far beyond this, however, to the "semisimple + nilpotent" decomposition of an arbitrary operator, and then on to the Jordan form and its real analogue. Those proofs that are far removed from our use of the theorems are relegated to appendices. While complex spaces are used freely, our primary concern is to obtain results for real spaces. This point of view, so important for differential equations, is not commonly found in textbooks on linear algebra or on differential equations.

Our approach to linear algebra is a fairly intrinsic one; we avoid coordinates where feasible, while not hesitating to use them as a tool for computations or proofs. On the other hand, instead of developing abstract vector spaces, we work with linear subspaces of $\mathbf{R}^n$ or $\mathbf{C}^n$, a small concession which perhaps makes the abstraction more digestible.

Using our algebraic theory, we give explicit methods of writing down solutions to arbitrary constant coefficient linear differential equations. Examples are included. In particular, the $S + N$ decomposition is used to compute the exponential of an arbitrary square matrix.

Chapter 2 is independent from the others and includes an elementary account of the Keplerian planetary orbits.

The fundamental theorems on existence, uniqueness, and continuity of solutions of ordinary differential equations are developed in Chapters 8 and 16. Chapter 8 is restricted to the autonomous case, in line with our basic orientation toward dynamical systems.

Chapters 10, 12, and 14 are devoted to systematic introductions to mathematical models of electrical circuits, population theory, and classical mechanics, respectively. The Brayton–Moser circuit theory is presented as a special case of the more general theory recently developed on manifolds. The Volterra–Lotka equations of competing species are analyzed, along with some generalizations. In mechanics we develop the Hamiltonian formalism for conservative systems whose configuration space is an open subset of a vector space.

The remaining five chapters contain a substantial introduction to the phase portrait analysis of nonlinear autonomous systems. They include a discussion of "generic" properties of linear flows, Liapunov and structural stability, Poincaré–Bendixson theory, periodic attractors, and perturbations. We conclude with an Afterword which points the way toward manifolds.

The following remarks should help the reader decide on which chapters to read and in what order.

Chapters 1 and 2 are elementary, but they present many ideas that recur throughout the book.

Chapters 3–7 form a sequence that develops linear theory rather thoroughly. Chapters 3, 4, and 5 make a good introduction to linear operators and linear differential equations. The canonical form theory of Chapter 6 is the basis of the stability results proved in Chapters 7, 9, and 13; however, this heavy algebra might be postponed at a first exposure to this material and the results taken on faith.

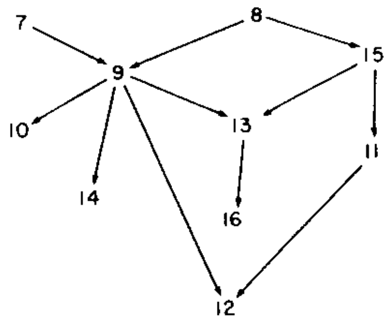
The existence, uniqueness, and continuity of solutions, proved in Chapter 8, are used (often implicitly) throughout the rest of the book. Depending on the reader's taste, proofs could be omitted.

A reader interested in the nonlinear material, who has some background in linear theory, might start with the stability theory of Chapter 9. Chapters 12 (ecology), 13 (periodic attractors), and 16 (perturbations) depend strongly on Chapter 9, while the section on dual vector spaces and gradients will make Chapters 10 (electrical circuits) and 14 (mechanics) easier to understand.

Chapter 12 also depends on Chapter 11 (Poincaré–Bendixson); and the material in Section 2 of Chapter 11 on local sections is used again in Chapters 13 and 16.

Chapter 15 (nonautonomous equations) is a continuation of Chapter 8 and is used in Chapters 11, 13, and 16; however it can be omitted at a first reading.

The logical dependence of the later chapters is summarized in the following chart:



The book owes much to many people. We only mention four of them here. Ikuko Workman and Ruth Suzuki did an excellent job of typing the manuscript. Dick Palais made a number of useful comments. Special thanks are due to Jacob Palis, who read the manuscript thoroughly, found many minor errors, and suggested several substantial improvements. Professor Hirsch is grateful to the Miller Institute for its support during part of the writing of the book.

Chapter 1

First Examples

The purpose of this short chapter is to develop some simple examples of differential equations. This development motivates the linear algebra treated subsequently and moreover gives in an elementary context some of the basic ideas of ordinary differential equations. Later these ideas will be put into a more systematic exposition. In particular, the examples themselves are special cases of the class of differential equations considered in Chapter 3. We regard this chapter as important since some of the most basic ideas of differential equations are seen in simple form.

§1. The Simplest Examples

The differential equation

$$\frac{dx}{dt} = ax \tag{1}$$

is the simplest differential equation. It is also one of the most important. First, what does it mean? Here $x = x(t)$ is an unknown real-valued function of a real variable t and dx/dt is its derivative (we will also use x' or $x'(t)$ for this derivative). The equation tells us that for every value of t the equality

$$x'(t) = ax(t)$$

is true. Here a denotes a constant.

The solutions to (1) are obtained from calculus: if K is any constant (real number), the function $f(t) = Ke^{at}$ is a solution since

$$f'(t) = aKe^{at} = af(t).$$

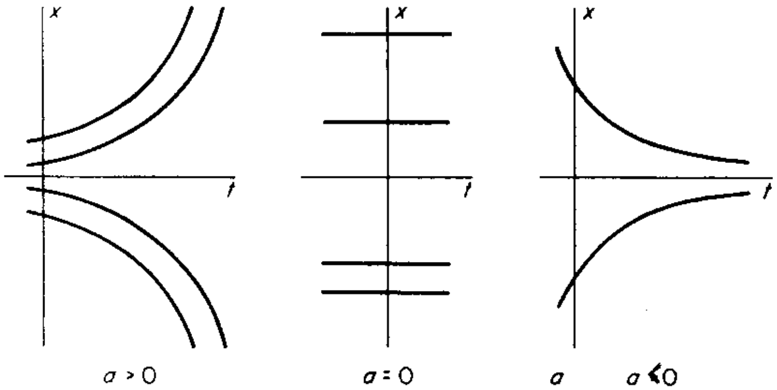


FIG. A

Moreover, *there are no other solutions*. To see this, let $u(t)$ be any solution and compute the derivative of $u(t)e^{-at}$:

$$\frac{d}{dt}\left(u(t)e^{-at}\right) = u'(t)e^{-at} + u(t)(-ae^{-at})$$

$$= au(t)e^{-at} - au(t)e^{-at} = 0.$$

Therefore $u(t)e^{-at}$ is a constant K, so $u(t) = Ke^{at}$. This proves our assertion.

The constant K appearing in the solution is completely determined if the value u_0 of the solution at a single point t_0 is specified. Suppose that a function $x(t)$ satisfying (1) is required such that $x(t_0) = u_0$, then K must satisfy $Ke^{at_0} = u_0$. Thus equation (1) has a unique solution satisfying a specified *initial condition* $x(t_0) = u_0$. For simplicity, we often take $t_0 = 0$; then $K = u_0$. There is no loss of generality in taking $t_0 = 0$, for if $u(t)$ is a solution with $u(0) = u_0$, then the function $v(t) = u(t - t_0)$ is a solution with $v(t_0) = u_0$.

It is common to restate (1) in the form of an *initial value problem*:

(2) $$x' = ax, \qquad x(0) = K.$$

A solution $x(t)$ to (2) must not only satisfy the first condition (1), but must also take on the prescribed initial value K at $t = 0$. We have proved that the initial value problem (2) has a unique solution.

The constant a in the equation $x' = ax$ can be considered as a parameter. If a changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change?

The *sign* of a is crucial here:

if $a > 0$, $\lim_{t\to\infty} Ke^{at}$ equals ∞ when $K > 0$, and equals $-\infty$ when $K < 0$;
if $a = 0$, $Ke^{at} =$ constant;
if $a < 0$, $\lim_{t\to\infty} Ke^{at} = 0$.

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions (Fig. A). These graphs follow a typical practice in this book. The figures are meant to illustrate qualitative features and may be imprecise in quantitative detail.

The equation $x' = ax$ is *stable* in a certain sense if $a \neq 0$. More precisely, if a is replaced by another constant b sufficiently close to a, the qualitative behavior of the solutions does not change. If, for example, $|b - a| < |a|$, then b has the same sign as a. But if $a = 0$, the slightest change in a leads to a radical change in the behavior of solutions. We may also say that $a = 0$ is a *bifurcation point* in the one-parameter family of equations $x' = ax$, a in $\mathbf{R}$.

Consider next a *system* of two differential equations in two unknown functions:

(3) $$x_1' = a_1x_1,$$
$$x_2' = a_2x_2.$$

This is a very simple system; however, many more-complicated systems of two equations can be reduced to this form as we shall see a little later.

Since there is no relation specified between the two unknown functions $x_1(t)$, $x_2(t)$, they are "uncoupled"; we can immediately write down all solutions (as for (1)):

$$x_1(t) = K_1\exp(a_1t), \qquad K_1 = \text{constant},$$
$$x_2(t) = K_2\exp(a_2t), \qquad K_2 = \text{constant}.$$

Here K_1 and K_2 are determined if initial conditions $x_1(t_0) = u_1$, $x_2(t_0) = u_2$ are specified. (We sometimes write exp a for e^a.)

Let us consider equation (2) from a more geometric point of view. We consider two functions $x_1(t)$, $x_2(t)$ as specifying an unknown *curve* $x(t) = (x_1(t), x_2(t))$ in the (x_1, x_2) plane $\mathbf{R}^2$. That is to say, x is a map from the real numbers $\mathbf{R}$ into $\mathbf{R}^2$, $x: \mathbf{R} \to \mathbf{R}^2$. The right-hand side of (3) expresses the *tangent vector* $x'(t) = (x_1'(t), x_2'(t))$ to the curve. Using vector notation,

(3′) $$x' = Ax,$$

where Ax denotes the vector (a_1x_1, a_2x_2), which one should think of as being based at x.

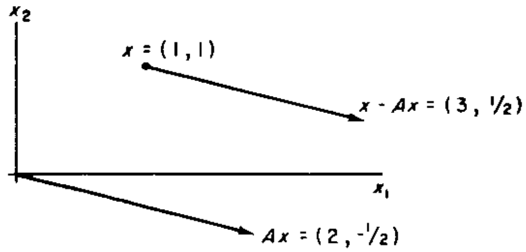


FIG. B

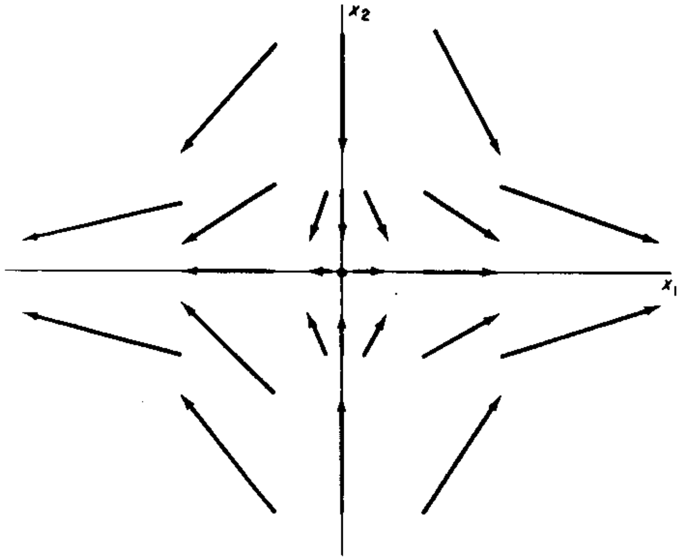


FIG. C. $Ax = (2x_1, -\frac{1}{2}x_2)$.

Initial conditions are of the form $x(t_0) = u$ where $u = (u_1, u_2)$ is a given point of $\mathbf{R}^2$. Geometrically, this means that when $t = t_0$ the curve is required to pass through the given point u.

The map (that is, function) $A: \mathbf{R}^2 \to \mathbf{R}^2$ (or $x \to Ax$) can be considered a *vector field* on $\mathbf{R}^2$. This means that to each point x in the plane we assign the vector Ax. For purposes of visualization, we picture Ax as a vector "based at x"; that is, we assign to x the directed line segment from x to $x + Ax$. For example, if $a_1 = 2$, $a_2 = -\frac{1}{2}$, and $x = (1, 1)$, then at $(1, 1)$ we picture an arrow pointing from $(1, 1)$ to $(1, 1) + (2, -\frac{1}{2}) = (3, \frac{1}{2})$ (Fig. B). Thus if $Ax = (2x_1, -\frac{1}{2}x_2)$, we attach to each point x in the plane an arrow with tail at x and head at $x + Ax$ and obtain the picture in Fig. C.

Solving the differential equation (3) or (3′) with initial conditions (u_1, u_2) at $t = 0$ means finding in the plane a curve $x(t)$ that satisfies (3′) and passes through the point $u = (u_1, u_2)$ when $t = 0$. A few solution curves are sketched in Fig. D. The trivial solution $(x_1(t), x_2(t)) = (0, 0)$ is also considered a "curve."

The family of all solution curves as subsets of $\mathbf{R}^2$ is called the "phase portrait" of equation (3) (or (3′)).

The one-dimensional equation $x' = ax$ can also be interpreted geometrically: the phase portrait is as in Fig. E, which should be compared with Fig. A. It is clearer to picture the graphs of (1) and the solution curves for (3) since two-dimensional pictures are better than either one- or three-dimensional pictures. The *graphs* of

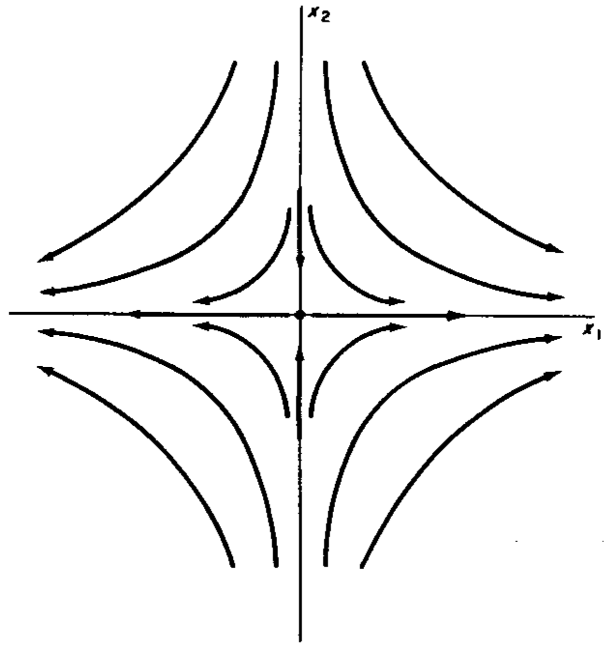


FIG. D. Some solution curves to $x' = Ax$, $A = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

solutions to (3) require a three-dimensional picture which the reader is invited to sketch!

Let us consider equation (3) as a *dynamical system*. This means that the independent variable t is interpreted as *time* and the solution curve $x(t)$ could be thought of, for example, as the path of a particle moving in the plane $\mathbf{R}^2$. We can imagine a particle placed at any point $u = (u_1, u_2)$ in $\mathbf{R}^2$ at time $t = 0$. As time proceeds the particle moves along the solution curve $x(t)$ that satisfies the initial condition $x(0) = u$. At any later time $t > 0$ the particle will be in another position $x(t)$. And at an earlier time $t < 0$, the particle was at a position $x(t)$. To indicate the dependence of the position on t and u we denote it by $\phi_t(u)$. Thus

$$\phi_t(u) = (u_1 \exp(a_1 t), u_2 \exp(a_2 t)).$$

We can imagine particles placed at each point of the plane and all moving simultaneously (for example, dust particles under a steady wind). The solution curves are spoken of as trajectories or orbits in this context. For each fixed t in $\mathbf{R}$, we have a transformation assigning to each point u in the plane another point $\phi_t(u)$. This transformation denoted by $\phi_t: \mathbf{R}^2 \to \mathbf{R}^2$ is clearly a *linear* transformation, that is,

FIG. E

$\phi_t(u + v) = \phi_t(u) + \phi_t(v)$ and $\phi_t(\lambda u) = \lambda \phi_t(u)$, for all vectors u, v, and all real numbers λ.

As time proceeds, every point of the plane moves simultaneously along the trajectory passing through it. In this way the collection of maps $\phi_t: \mathbf{R}^2 \to \mathbf{R}^2$, $t \in \mathbf{R}$, is a one-parameter family of transformations. This family is called the *flow* or *dynamical system* or $\mathbf{R}^2$ determined by the vector field $x \to Ax$, which in turn is equivalent to the system (3).

The dynamical system on the real line $\mathbf{R}$ corresponding to equation (1) is particularly easy to describe: if $a < 0$, all points move toward 0 as time goes to ∞; if $a > 0$, all points except 0 move away from 0 toward $\pm\infty$; if $a = 0$, all points stand still.

We have started from a differential equation and have obtained the dynamical system ϕ_t. This process is established through the fundamental theorem of ordinary differential equations as we shall see in Chapter 8.

Later we shall also reverse this process: starting from a dynamical system ϕ_t, a differential equation will be obtained (simply by differentiating $\phi_t(u)$ with respect to t).

It is seldom that differential equations are given in the simple uncoupled form (3). Consider, for example, the system:

(4)
$$x_1' = 5x_1 + 3x_2,$$
$$x_2' = -6x_1 - 4x_2$$

or in vector notation

(4′)
$$x' = (5x_1 + 3x_2, -6x_1 - 4x_2) \equiv Bx.$$

Our approach is to find a linear *change of coordinates* that will transform equation (4) into uncoupled or diagonal form. It turns out that new coordinates (y_1, y_2) do the job where

$$y_1 = 2x_1 + x_2,$$
$$y_2 = x_1 + x_2.$$

(In Chapter 3 we explain how the new coordinates were found.)

Solving for x in terms of y, we have

$$x_1 = y_1 - y_2,$$
$$x_2 = -y_1 + 2y_2.$$

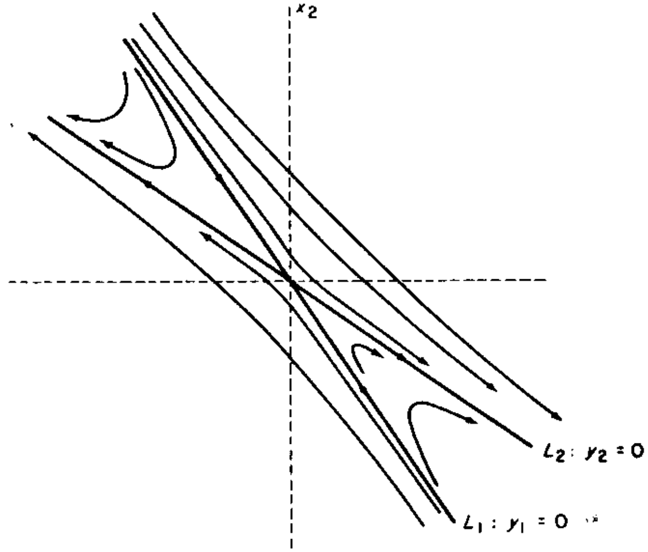


FIG. F

To find y_1', y_2' differentiate the equations defining y_1, y_2 to obtain

$$y_1' = 2x_1' + x_2',$$
$$y_2' = x_1' + x_2'.$$

By substitution

$$y_1' = 2(5x_1 + 3x_2) + (-6x_1 - 4x_2) = 4x_1 + 2x_2,$$
$$y_2' = (5x_1 + 3x_2) + (-6x_1 - 4x_2) = -x_1 - x_2.$$

Another substitution yields

$$y_1' = 4(y_1 - y_2) + 2(-y_1 + 2y_2),$$
$$y_2' = -(y_1 - y_2) - (-y_1 + 2y_2),$$

or

(5)
$$y_1' = 2y_1,$$
$$y_2' = -y_2.$$

The last equations are in *diagonal form* and we have already solved this class of systems. The solution $(y_1(t), y_2(t))$ such that $(y_1(0), y_2(0)) = (v_1, v_2)$ is

$$y_1(t) = e^{2t}v_1,$$
$$y_2(t) = e^{-t}v_2.$$

The phase portrait of this system (5) is given evidently in Fig. D. We can find the phase portrait of the original system (4) by simply plotting the new coordinate axes $y_1 = 0$, $y_2 = 0$ in the (x_1, x_2) plane and sketching the trajectories $y(t)$ in these coordinates. Thus $y_1 = 0$ is the line L_1: $x_2 = -2x_1$ and $y_2 = 0$ is the line L_2: $x_2 = -x_1$.

Thus we have the phase portrait of (4) as in Fig. F, which should be compared with Fig. D.

Formulas for the solution to (4) can be obtained by substitution as follows. Let (u_1, u_2) be the initial values $(x_1(0), x_2(0))$ of a solution $(x_1(t), x_2(t))$ to (4). Corresponding to (u_1, u_2) is the initial value (v_1, v_2) of a solution $(y_1(t), y_2(t))$ to (5) where

$$v_1 = 2u_1 + u_2,$$
$$v_2 = u_1 + u_2.$$

Thus

$$y_1(t) = e^{2t}(2u_1 + u_2),$$
$$y_2(t) = e^{-t}(u_1 + u_2)$$

and

$$x_1(t) = e^{2t}(2u_1 + u_2) - e^{-t}(u_1 + u_2),$$
$$x_2(t) = -e^{2t}(2u_1 + u_2) + 2e^{-t}(u_1 + u_2).$$

If we compare these formulas to Fig. F, we see that the diagram instantly gives us the qualitative picture of the solutions, while the formulas convey little geometric information. In fact, for many purposes, it is better to forget the original equation (4) and the corresponding solutions and work entirely with the "diagonalized" equations (5), their solution and phase portrait.

PROBLEMS

1. Each of the "matrices"

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [a_{ij}]$$

given below defines a vector field on $\mathbf{R}^2$, assigning to $x = (x_1, x_2) \in \mathbf{R}^2$ the vector $Ax = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$ based at x. For each matrix, draw enough of the vectors until you get a feeling for what the vector field looks like. Then sketch the phase portrait of the corresponding differential equation $x' = Ax$, guessing where necessary.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & -2 \\ 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ (h) $\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ (i) $\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$

2. Consider the one-parameter family of differential equations

$$x_1' = 2x_1,$$
$$x_2' = ax_2; \qquad -\infty < a < \infty.$$

(a) Find all solutions $(x_1(t), x_2(t))$.
(b) Sketch the phase portrait for a equal to $-1, 0, 1, 2, 3$. Make some guesses about the stability of the phase portraits.

§2. Linear Systems with Constant Coefficients

This section is devoted to generalizing and abstracting the previous examples. The general problem is stated, but solutions are postponed to Chapter 3.

Consider the following set or "system" of n differential equations:

$$(1) \qquad \begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

Here the a_{ij} $(i = 1, \ldots, n; j = 1, \ldots, n)$ are n^2 constants (real numbers), while each x_i denotes an unknown real-valued function of a real variable t. Thus (4) of Section 1 is an example of the system (1) with $n = 2$, $a_{11} = 5$, $a_{12} = 3$, $a_{21} = -6$, $a_{22} = -4$.

At this point we are not trying to solve (1); rather, we want to place it in a geometrical and algebraic setting in order to understand better what a solution means.

At the most primitive level, a solution of (1) is a set of n differentiable real-valued functions $x_i(t)$ that make (1) true.

In order to reach a more conceptual understanding of (1) we introduce *real n-dimensional Cartesian space* $\mathbf{R}^n$. This is simply the set of all n-tuples of real numbers. An element of $\mathbf{R}^n$ is a "point" $x = (x_1, \ldots, x_n)$; the number x_i is the ith *coordinate* of the point x. Points x, y in $\mathbf{R}^n$ are added coordinatewise:

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Also, if λ is a real number we define the *product* of λ and x to be

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n).$$

The *distance* between points x, y in $\mathbf{R}^n$ is defined to be

$$|x - y| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

The *length* of x is

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

A vector *based at* $x \in \mathbf{R}^n$ is an ordered pair of points x, y in $\mathbf{R}^n$, denoted by $\overrightarrow{xy}$. We think of this as an arrow or line segment directed from x to y. We say $\overrightarrow{xy}$ is *based* at x.

A vector $\overrightarrow{0x}$ based at the *origin*

$$0 = (0, \ldots, 0) \in \mathbf{R}^n$$

is identified with the point $x \in \mathbf{R}^n$.

To a vector $\overrightarrow{xy}$ based at x is associated the vector $y - x$ based at the origin 0. We call the vectors $\overrightarrow{xy}$ and $y - x$ *translates* of each other.

From now on a vector based at 0 is called simply a vector. Thus an element of $\mathbf{R}^n$ can be considered either as an n-tuple of real numbers or as an arrow issuing from the origin.

It is only for purposes of visualization that we consider vectors based at points other than 0. For computations, all vectors are based at 0 since such vectors can be added and multiplied by real numbers.

We return to the system of differential equations (1). A candidate for a solution is a *curve* in $\mathbf{R}^n$:

$$(*) \qquad x(t) = (x_1(t), \ldots, x_n(t)).$$

By this we mean a map

$$x\colon \mathbf{R} \to \mathbf{R}^n.$$

Such a map is described in terms of coordinates by $(*)$. If each function $x_i(t)$ is differentiable, then the map x is called differentiable; its derivative is defined to be

$$\frac{dx}{dt} = x'(t) = (x_1'(t), \ldots, x_n'(t)).$$

Thus the derivative, as a function of t, is again a map from $\mathbf{R}$ to $\mathbf{R}^n$.

The derivative can also be expressed in the form

$$x'(t) = \lim_{h \to 0} \frac{1}{h}(x(t + h) - x(t)).$$

It has a natural geometric interpretation as the vector $v(t)$ based at $x(t)$, which is a translate of $x'(t)$. This vector is called the *tangent vector* to the curve at t (or at $x(t)$).

If we imagine t as denoting time, then the length $|x'(t)|$ of the tangent vector is interpreted physically as the speed of a particle describing the curve $x(t)$.

To write (1) in an abbreviated form we call the doubly indexed set of numbers a_{ij} an $n \times n$ *matrix* A, denoted thus:

$$A = [a_{ji}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Next, for each $x \in \mathbf{R}^n$ we define a vector $Ax \in \mathbf{R}^n$ whose ith coordinate is

$$a_{i1}x_1 + \cdots + a_{in}x_n;$$

note that this is the ith row in the right-hand side of (1). In this way the matrix A is interpreted as a map

$$A\colon \mathbf{R}^n \to \mathbf{R}^n$$

which to x assigns Ax.

With this notation (1) is rewritten

$$(2) \qquad x' = Ax.$$

Thus the system (1) can be considered as a single "vector differential equation" (2). (The word *equation* is classically reserved for the case of just one variable; we shall call (2) both a system and an equation.)

We think of the map $A\colon \mathbf{R}^n \to \mathbf{R}^n$ as a *vector field* on $\mathbf{R}^n$: to each point $x \in \mathbf{R}^n$ it assigns the vector based at x which is a translate of Ax. Then a solution of (2) is a curve $x\colon \mathbf{R} \to \mathbf{R}^n$ whose tangent vector at any given t is the vector $Ax(t)$ (translated to $x(t)$). See Fig. D of Section 1.

In Chapters 3 and 4 we shall give methods of explicitly solving (2), or equivalently (1). In subsequent chapters it will be shown that in fact (2) has a unique solution $x(t)$ satisfying any given initial condition $x(0) = u_0 \in \mathbf{R}^n$. This is the fundamental theorem of linear differential equations with constant coefficients; in Section 1 this was proved for the special case $n = 1$.

PROBLEMS

1. For each of the following matrices A sketch the vector field $x \to Ax$ in $\mathbf{R}^3$. (Missing matrix entries are 0.)

(a) $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & & \\ & -2 & \\ & & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & & \\ & -2 & \\ & & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & & \\ & -1 & \\ & & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & -\frac{1}{2} \end{bmatrix}$ (f) $\begin{bmatrix} -1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$

2. For A as in (a), (b), (c) of Problem 1, solve the initial value problem

$$x' = Ax, \qquad x(0) = (k_1, k_2, k_3).$$

3. Let A be as in (e), Problem 1. Find constants a, b, c such that the curve $t \to (a \cos t, b \sin t, ce^{-t/2})$ is a solution to $x' = Ax$ with $x(0) = (1, 0, 3)$.

4. Find two different matrices A, B such that the curve

$$x(t) = (e^t, 2e^{2t}, 4e^{2t})$$

satisfies both the differential equations

$$x' = Ax \qquad \text{and} \qquad x' = Bx.$$

5. Let $A = [a_{ij}]$ be an $n \times n$ diagonal matrix, that is, $a_{ij} = 0$ if $i \neq j$. Show that the differential equation

$$x' = Ax$$

has a unique solution for every initial condition.

6. Let A be an $n \times n$ diagonal matrix. Find conditions on A guaranteeing that

$$\lim_{t \to \infty} x(t) = 0$$

for all solutions to $x' = Ax$.

7. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by $-A$ the matrix $[-a_{ij}]$.
   (a) What is the relation between the vector fields $x \to Ax$ and $x \to (-A)x$?
   (b) What is the geometric relation between solution curves of $x' = Ax$ and of $x' = -Ax$?

8. (a) Let $u(t)$, $v(t)$ be solutions to $x' = Ax$. Show that the curve $w(t) = \alpha u(t) + \beta v(t)$ is a solution for all real numbers α, β.

(b) Let $A = \begin{bmatrix} 1 & \\ & -2 \end{bmatrix}$. Find solutions $u(t)$, $v(t)$ to $x' = Ax$ such that every solution can be expressed in the form $\alpha u(t) + \beta v(t)$ for suitable constants α, β.

Notes

The background needed for a reader of Chapter 1 is a good first year of college calculus. One good source is S. Lang's *Second Course in Calculus* [12, Chapters I, II, and IX]. In this reference the material on derivatives, curves, and vectors in $\mathbf{R}^n$ and matrices is discussed much more thoroughly than in our Section 2.

Chapter 2

Newton's Equation and Kepler's Law

We develop in this chapter the earliest important examples of differential equations, which in fact are connected with the origins of calculus. These equations were used by Newton to derive and unify the three laws of Kepler. These laws were found from the earlier astronomical observations of Tycho Brahe. Here we give a brief derivation of two of Kepler's laws, while at the same time setting forth some general ideas about differential equations.

The equations of Newton, our starting point, have retained importance throughout the history of modern physics and lie at the root of that part of physics called classical mechanics.

The first chapter of this book dealt with linear equations, but Newton's equations are nonlinear in general. In later chapters we shall pursue the subject of nonlinear differential equations somewhat systematically. The examples here provide us with concrete examples of historical and scientific importance. Furthermore, the case we consider most thoroughly here, that of a particle moving in a central force gravitational field, is simple enough so that the differential equations can be solved explicitly using exact, classical methods (just calculus!). This is due to the existence of certain invariant functions called *integrals* (sometimes called "first integrals"; we do not mean the integrals of elementary calculus). Physically, an integral is a conservation law; in the case of Newtonian mechanics the two integrals we find correspond to conservation of energy and angular momentum. Mathematically an integral reduces the number of dimensions.

We shall be working with a particle moving in a *field of force* F. Mathematically F is a *vector field* on the (configuration) space of the particle, which in our case we suppose to be Cartesian three space $\mathbf{R}^3$. Thus F is a map $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ that assigns to a point x in $\mathbf{R}^3$ another point $F(x)$ in $\mathbf{R}^3$. From the mathematical point of view, $F(x)$ is thought of as a vector based at x. From the physical point of view, $F(x)$ is the force exerted on a particle located at x.

The example of a force field we shall be most concerned with is the gravitational field of the sun: $F(x)$ is the force on a particle located at x attracting it to the sun.

We shall go into details of this field in Section 6. Other important examples of force fields are derived from electrical forces, magnetic forces, and so on.

The connection between the physical concept of force field and the mathematical concept of differential equation is *Newton's second law*: $F = ma$. This law asserts that a particle in a force field moves in such a way that the force vector at the location of the particle, at any instant, equals the acceleration vector of the particle times the mass m. If $x(t)$ denotes the position vector of the particle at time t, where $x\colon \mathbf{R} \to \mathbf{R}^3$ is a sufficiently differentiable curve, then the acceleration vector is the second derivative of $x(t)$ with respect to time

$$a(t) = \ddot{x}(t).$$

(We follow tradition and use dots for time derivatives in this chapter.) Newton's second law states

$$F(x(t)) = m\ddot{x}(t).$$

Thus we obtain a second order differential equation:

$$\ddot{x} = \frac{1}{m} F(x).$$

In Newtonian physics it is assumed that m is a positive constant. Newton's law of gravitation is used to derive the exact form of the function $F(x)$. While these equations are the main goal of this chapter, we first discuss simple harmonic motion and then basic background material.

§1. Harmonic Oscillators

We consider a particle of mass m moving in one dimension, its position at time t given by a function $t \to x(t)$, $x\colon \mathbf{R} \to \mathbf{R}$. Suppose the force on the particle at a point $x \in \mathbf{R}$ is given by $-mp^2x$, where p is some real constant. Then according to the laws of physics (compare Section 3) the motion of the particle satisfies

$$\ddot{x} + p^2x = 0. \tag{1}$$

This model is called the harmonic oscillator and (1) is the equation of the harmonic oscillator (in one dimension).

An example of the harmonic oscillator is the simple pendulum moving in a plane, when one makes an approximation of $\sin x$ by x (compare Chapter 9). Another example is the case where the force on the particle is caused by a spring.

It is easy to check that for any constants A, B, the function

$$x(t) = A \cos pt + B \sin pt \tag{2}$$

is a solution of (1), with initial conditions $x(0) = A$, $\dot{x}(0) = pB$. In fact, as is proved often in calculus courses, (2) is the only solution of (1) satisfying these initial conditions. Later we will show in a systematic way that these facts are true.

Using basic trigonometric identities, (2) may be rewritten in the form

$$x(t) = a\cos(pt + t_0), \tag{3}$$

where $a = (A^2 + B^2)^{1/2}$ is called the amplitude, and $\cos t_0 = A(A^2 + B^2)^{-1/2}$.

In Section 6 we will consider equation (1) where a constant term is added (representing a constant disturbing force):

$$\ddot{x} + p^2 x = K. \tag{4}$$

Then, similarly to (1), every solution of (4) has the form

$$x(t) = a\cos(pt + t_0) + \frac{K}{p^2}. \tag{5}$$

The two-dimensional version of the harmonic oscillator concerns a map $x\colon \mathbf{R} \to \mathbf{R}^2$ and a force $F(x) = -mkx$ (where now, of course, $x = (x_1, x_2) \in \mathbf{R}^2$). Equation (1) now has the same form

$$\ddot{x} + k^2 x = 0 \tag{1'}$$

with solutions given by

$$x_1(t) = A\cos kt + B\sin kt, \tag{2'}$$

$$x_2(t) = C\cos kt + D\sin kt.$$

See Problem 1.

Planar motion will be considered more generally and in more detail in later sections. But first we go over some mathematical preliminaries.

§2. Some Calculus Background

A path of a moving particle in $\mathbf{R}^n$ (usually $n \le 3$) is given by a map $f\colon I \to \mathbf{R}^n$ where I might be the set $\mathbf{R}$ of all real numbers or an interval (a, b) of all real numbers strictly between a and b. The derivative of f (provided f is differentiable at each point of I) defines a map $f'\colon I \to \mathbf{R}^n$. The map f is called C^1, or *continuously differentiable*, if f' is continuous (that is to say, the corresponding coordinate functions $f_i'(t)$ are continuous, $i = 1, \ldots, n$). If $f'\colon I \to \mathbf{R}^n$ is itself C^1, then f is said to be C^2. Inductively, in this way, one defines a map $f\colon I \to \mathbf{R}^n$ to be C^r, where $r = 3, 4, 5$, and so on.

The *inner product*, or "dot product," of two vectors, x, y in $\mathbf{R}^n$ is denoted by $\langle x, y\rangle$ and defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

Thus $\langle x, x\rangle = |x|^2$. If $x, y\colon I \to \mathbf{R}^n$ are C^1 functions, then a version of the Leibniz product rule for derivatives is

$$\langle x, y\rangle' = \langle x', y\rangle + \langle x, y'\rangle,$$

as can be easily checked using coordinate functions.

We will have occasion to consider functions $f\colon \mathbf{R}^n \to \mathbf{R}$ (which, for example, could be given by temperature or density). Such a map f is called C^1 if the map $\mathbf{R}^n \to \mathbf{R}$ given by each partial derivative $x \to \partial f/\partial x_i(x)$ is defined and continuous (in Chapter 5 we discuss continuity in more detail). In this case the gradient of f, called *grad* f, is the map $\mathbf{R}^n \to \mathbf{R}^n$ that sends x into $(\partial f/\partial x_1(x), \ldots, \partial f/\partial x_n(x))$. Grad f is an example of a vector field on $\mathbf{R}^n$. (In Chapter 1 we considered only linear vector fields, but grad f may be more general.)

Next, consider the composition of two C^1 maps as follows:

$$I \xrightarrow{f} \mathbf{R}^n \xrightarrow{g} \mathbf{R}.$$

The chain rule can be expressed in this context as

$$\frac{d}{dt}g(f(t)) = \langle \operatorname{grad} g(f(t)), f'(t)\rangle;$$

using the definitions of gradient and inner product, the reader can prove that this is equivalent to

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(f(t))\frac{df_i}{dt}(t).$$

§3. Conservative Force Fields

A vector field $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ is called a force field if the vector $F(x)$ assigned to the point x is interpreted as a force acting on a particle placed at x.

Many force fields appearing in physics arise in the following way. There is a C^1 function

$$V\colon \mathbf{R}^3 \to \mathbf{R}$$

such that

$$F(x) = -\left(\frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x), \frac{\partial V}{\partial x_3}(x)\right)$$

$$= -\operatorname{grad} V(x).$$

(The negative sign is traditional.) Such a force field is called *conservative*. The function V is called the *potential energy* function. (More properly V should be called *a* potential energy since adding a constant to it does not change the force field $-\operatorname{grad} V(x)$.) Problem 4 relates potential energy to *work*.

The planar harmonic oscillation of Section 1 corresponds to the force field

$$F: \mathbf{R}^2 \to \mathbf{R}^2, \qquad F(x) = -mkx.$$

This field is conservative, with potential energy

$$V(x) = \tfrac{1}{2}mk \mid x \mid^2$$

as is easily verified.

For any moving particle $x(t)$ of mass m, the *kinetic energy* is defined to be

$$T = \tfrac{1}{2}m \mid \dot{x}(t) \mid^2.$$

Here $\dot{x}(t)$ is interpreted as the *velocity vector* at time t; its length $\mid \dot{x}(t) \mid$ is the *speed* at time t. If we consider the function $x: \mathbf{R} \to \mathbf{R}^3$ as describing a curve in $\mathbf{R}^3$, then $\dot{x}(t)$ is the *tangent vector* to the curve at $x(t)$.

For a particle moving in a conservative force field $F = -\text{grad } V$, the potential energy at x is defined to be $V(x)$. Note that whereas the kinetic energy depends on the velocity, the potential energy is a function of position.

The *total energy* (or sometimes simply *energy*) is

$$E = T + V.$$

This has the following meaning. If $x(t)$ is the trajectory of a particle moving in the conservative force field, then E is a real-valued function of time:

$$E(t) = \tfrac{1}{2} \mid m\dot{x}(t) \mid^2 + V(x(t)).$$

Theorem (Conservation of Energy) *Let $x(t)$ be the trajectory of a particle moving in a conservative force field $F = -\text{grad } V$. Then the total energy E is independent of time.*

Proof. It needs to be shown that $E(x(t))$ is constant in t or that

$$\frac{d}{dt}(T + V) = 0,$$

or equivalently,

$$\frac{d}{dt}\left(\frac{1}{2} m \mid \dot{x}(t) \mid^2 + V(x(t))\right) = 0.$$

It follows from calculus that

$$\frac{d}{dt} \mid \dot{x} \mid^2 = 2\langle \ddot{x}, \dot{x} \rangle$$

(a version of the Leibniz product formula); and also that

$$\frac{d}{dt}(V(\dot{x})) = \langle \text{grad } V(x), \dot{x} \rangle$$

(the chain rule).

These facts reduce the proof to showing that

$$m\langle \ddot{x}, \dot{x} \rangle + \langle \text{grad } V, \dot{x} \rangle = 0$$

or $\langle m\ddot{x} + \text{grad } V, \dot{x} \rangle = 0$. But this is so since Newton's second law is $m\ddot{x} + \text{grad } V(x) = 0$ in this instance.

§4. Central Force Fields

A force field F is called *central* if $F(x)$ points in the direction of the line through x, for every x. In other words, the vector $F(x)$ is always a scalar multiple of x, the coefficient depending on x:

$$F(x) = \lambda(x)x.$$

We often tacitly exclude from consideration a particle at the origin; many central force fields are not defined (or are "infinite") at the origin.

Lemma *Let F be a conservative force field. Then the following statements are equivalent:*

(a) *F is central,*
(b) *$F(x) = f(\mid x \mid)x$,*
(c) *$F(x) = -\text{grad } V(x)$ and $V(x) = g(\mid x \mid)$.*

Proof. Suppose (c) is true. To prove (b) we find, from the chain rule:

$$\frac{\partial V}{\partial x_j} = g'(\mid x \mid) \frac{\partial}{\partial x_j}(x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$= \frac{g'(\mid x \mid)}{\mid x \mid} x_j;$$

this proves (b) with $f(\mid x \mid) = g'(\mid x \mid)/\mid x \mid$. It is clear that (b) implies (a). To show that (a) implies (c) we must prove that V is constant on each sphere.

$$S_\alpha = \{x \in \mathbf{R}^3 \mid \mid x \mid = \alpha\}, \qquad \alpha > 0.$$

Since any two points in S_α can be connected by a curve in S_α, it suffices to show that V is constant on any curve in S_α. Hence if $J \subset \mathbf{R}$ is an interval and $u: J \to S_\alpha$ is a C^1 map, we must show that the derivative of the composition $V \circ u$

$$J \xrightarrow{u} S_\alpha \subset \mathbf{R}^3 \xrightarrow{V} \mathbf{R}$$

is identically 0. This derivative is

$$\frac{d}{dt} V(u(t)) = \langle \text{grad } V(u(t)), u'(t) \rangle$$

as in Section 2. Now grad $V(x) = -F(x) = -\lambda(x)x$ since F is central:

$$\frac{d}{dt}V(u(t)) = -\lambda(u(t))\langle u(t), u'(t)\rangle$$

$$= \frac{-\lambda u(t))}{2}\frac{d}{dt}|u(t)|^2$$

$$= 0$$

because $|u(t)| \equiv \alpha$.

In Section 5 we shall consider a special conservative central force field obtained from Newton's law of gravitation.

Consider now a central force field, not necessarily conservative.

Suppose at some time t_0, that $P \subset \mathbf{R}^3$ denotes the plane containing the particle, the velocity vector of the particle and the origin. The force vector $F(x)$ for any point x in P also lies in P. This makes it plausible that the particle stays in the plane P for all time. In fact, this is true: a particle moving in a central force field moves in a fixed plane.

The proof depends on the *cross product* (or vector product) $u \times v$ of vectors u, v in $\mathbf{R}^3$. We recall the definition

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \in \mathbf{R}^3$$

and that $u \times v = -v \times u = |u||v| N \sin\theta$, where N is a unit vector perpendicular to u and v, (U, v, N) oriented as the axes ("right-hand rule"), and θ is the angle between u and v.

Then the vector $u \times v = 0$ if and only if one vector is a scalar multiple of the other; if $u \times v \neq 0$, then $u \times v$ is orthogonal to the plane containing u and v. If u and v are functions of t in $\mathbf{R}$, then a version of the Leibniz product rule asserts (as one can check using Cartesian coordinates):

$$\frac{d}{dt}(u \times v) = \dot{u} \times v + u \times \dot{v}.$$

Now let $x(t)$ be the path of a particle moving under the influence of a central force field. We have

$$\frac{d}{dt}(x \times \dot{x}) = \dot{x} \times \dot{x} + x \times \ddot{x}$$

$$= x \times \ddot{x}$$

$$= 0$$

because $\ddot{x}$ is a scalar multiple of x. Therefore $x(t) \times \dot{x}(t)$ is a constant vector y. If $y \neq 0$, this means that x and $\dot{x}$ always lie in the plane orthogonal to y, as asserted. If $y = 0$, then $\dot{x}(t) = g(t)x(t)$ for some scalar function $g(t)$. This means that the velocity vector of the moving particle is always directed along the line through the origin and the particle, as is the force on the particle. This makes it plausible that the particle always moves along the same line through the origin. To prove this let $(x_1(t), x_2(t), x_3(t))$ be the coordinates of $x(t)$. Then we have three differential equations

$$\frac{dx_k}{dt} = g(t)x_k(t), \qquad k = 1, 2, 3.$$

By integration we find

$$x_k(t) = e^{h(t)}x_k(t_0), \qquad h(t) = \int_{t_0}^{t} g(s)\,ds.$$

Therefore $x(t)$ is always a scalar multiple of $x(t_0)$ and so $x(t)$ moves in a fixed line, and hence in a fixed plane, as asserted.

We restrict attention to a conservative central force field in a plane, which we take to be the Cartesian plane $\mathbf{R}^2$. Thus x now denotes a point of $\mathbf{R}^2$, the potential energy V is defined on $\mathbf{R}^2$ and

$$F(x) = -\text{grad } V(x) = -\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right).$$

Introduce polar coordinates (r, θ), with $r = |x|$.

Define the *angular momentum* of the particle to be

$$h = mr^2\dot{\theta},$$

where $\dot{\theta}$ is the time derivative of the angular coordinate of the particle.

Theorem (Conservation of Angular Momentum) *For a particle moving in a central force field:*

$$\frac{dh}{dt} = 0, \qquad \textit{where} \quad h = mr^2\dot{\theta}.$$

Proof. Let $i = i(t)$ be the unit vector in the direction $x(t)$ so $x = ri$. Let $j = j(t)$ be the unit vector with a 90° angle from i to j. A computation shows that $di/dt = \dot{\theta}j$, $dj/dt = -\dot{\theta}i$ and hence

$$\dot{x} = \dot{r}i + r\dot{\theta}j.$$

Differentiating again yields

$$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)i + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})j.$$

If the force is central, however, it has zero component perpendicular to x. Therefore, since $\ddot{x} = m^{-1}F(x)$, the component of $\ddot{x}$ along j must be 0. Hence

$$\frac{d}{dt}(r^2\dot{\theta}) = 0,$$

proving the theorem.

We can now prove one of Kepler's laws. Let $A(t)$ denote the area swept out by the vector $x(t)$ in the time from t_0 to t. In polar coordinates $dA = \frac{1}{2}r^2\,d\theta$. We define the *areal velocity* to be

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta},$$

the rate at which the position vector sweeps out area. Kepler observed that the line segment joining a planet to the sun sweeps out equal areas in equal times, which we interpret to mean $\dot{A}$ = constant. We have proved more generally that this is true for any particle moving in a conservative central force field; this is a consequence of conservation of angular momentum.

§5. States

We recast the Newtonian formulation of the preceding sections in such a way that the differential equation becomes first order, the states of the system are made explicit, and energy becomes a function on the space of states.

A *state* of a physical system is information characterizing it at a given time. In particular, a state of the physical system of Section 1 is the position and velocity of the particle. The space of states is the Cartesian product $\mathbf{R}^3 \times \mathbf{R}^3$ of pairs (x, v), x, v in $\mathbf{R}^3$; x is the position, v the velocity that a particle might have at a given moment.

We may rewrite Newton's equation

$$m\ddot{x} = F(x) \tag{1}$$

as a first order equation in terms of x and v. (The *order* of a differential equation is the order of the highest derivative that occurs explicitly in the equation.) Consider the differential equation

$$\frac{dx}{dt} = v, \qquad m\frac{dv}{dt} = F(x). \tag{1'}$$

A solution to (1′) is a curve $t \to (x(t), v(t))$ in the state space $\mathbf{R}^3 \times \mathbf{R}^3$ such that

$$\dot{x}(t) = v(t) \qquad \text{and} \qquad \dot{v}(t) = m^{-1}F(x(t)) \qquad \text{for all } t.$$

It can be seen then that the solutions of (1) and (1′) correspond in a natural fashion. Thus if $x(t)$ is a solution of (1), we obtain a solution of (1′) by setting $v(t) = \dot{x}(t)$. *The map* $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3 \times \mathbf{R}^3$ *that sends* (x, v) *into* $(v, m^{-1}F(x))$ *is a vector field on the space of states, and this vector field defines the differential equation,* (1′).

A solution $(x(t), v(t))$ to (1′) gives the passage of the state of the system in time.

Now we may interpret energy as a function on the state space, $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$, defined by $E(x, v) = \frac{1}{2}m|v|^2 + V(x)$. The statement that "the energy is an integral" then means that the composite function

$$t \to (x(t), v(t)) \to E(x(t), v(t))$$

is constant, or that *on a solution curve in the state space, E is constant.*

We abbreviate $\mathbf{R}^3 \times \mathbf{R}^3$ by $\mathbb{S}$. An *integral* (for (1′)) on $\mathbb{S}$ is then any function that is constant on every solution curve of (1′). It was shown in Section 4 that in addition to energy, angular momentum is also an integral for (1′). In the nineteenth century, the idea of solving a differential equation was tied to the construction of a sufficient number of integrals. However, it is realized now that integrals do not exist for differential equations very generally; the problems of differential equations have been considerably freed from the need for integrals.

Finally, we observe that the force field may not be defined on all of $\mathbf{R}^3$, but only on some portion of it, for example, on an open subset $U \subset \mathbf{R}^3$. In this case the path $x(t)$ of the particle is assumed to lie in U. The force and velocity vectors, however, are still allowed to be arbitrary vectors in $\mathbf{R}^3$. The force field is then a vector field on U, denoted by $F: U \to \mathbf{R}^3$. The state space is the Cartesian product $U \times \mathbf{R}^3$, and (1′) is a first order equation on $U \times \mathbf{R}^3$.

§6. Elliptical Planetary Orbits

We now pass to consideration of Kepler's first law, that planets have elliptical orbits. For this, a central force is not sufficient. We need the precise form of V as given by the "inverse square law."

We shall show that in polar coordinates (r, θ), an orbit with nonzero angular momentum h is the set of points satisfying

$$r(1 + \epsilon\cos\theta) = l = \text{constant}; \qquad \epsilon = \text{constant},$$

which defines a conic, as can be seen by putting $r\cos\theta = x$, $r^2 = x^2 + y^2$.

Astronomical observations have shown the orbits of planets to be (approximately) ellipses.

Newton's law of gravitation states that a body of mass m_1 exerts a force on a body of mass m_2. The magnitude of the force is gm_1m_2/r^2, where r is the distance between their centers of gravity and g is a constant. The direction of the force on m_2 is from m_2 to m_1.

Thus if m_1 lies at the origin of $\mathbf{R}^3$ and m_2 lies at $x \in \mathbf{R}^3$, the force on m_2 is

$$-gm_1m_2\frac{x}{|x|^3}.$$

The force on m_1 is the negative of this.

We must now face the fact that *both* bodies will move. However, if m_1 is much greater than m_2, its motion will be much less since acceleration is inversely proportional to mass. We therefore make the simplifying assumption that one of the bodies does not move; in the case of planetary motion, of course it is the sun that is assumed at rest. (One might also proceed by taking the center of mass at the origin, without making this simplifying assumption.)

We place the sun at the origin of $\mathbf{R}^3$ and consider the force field corresponding to a planet of given mass m. This field is then

$$F(x) = -C\frac{x}{|x|^3},$$

where C is a constant. We then change the units in which force is measured to obtain the simpler formula

$$F(x) = -\frac{x}{|x|^3}.$$

It is clear this force field is central. Moreover, it is conservative, since

$$\frac{x}{|x|^3} = \text{grad } V,$$

where

$$V = \frac{-1}{|x|}.$$

Observe that $F(x)$ is not defined at 0.

As in the previous section we restrict attention to particles moving in the plane $\mathbf{R}^2$; or, more properly, in $\mathbf{R}^2 - 0$. The force field is the Newtonian gravitational field in $\mathbf{R}^2$, $F(x) = -x/|x|^3$.

Consider a particular solution curve of our differential equation $\ddot{x} = m^{-1}F(x)$. The angular momentum h and energy E are regarded as constants in time since they are the same at all points of the curve. The case $h = 0$ is not so interesting; it corresponds to motion along a straight line toward or away from the sun. Hence we assume $h \neq 0$.

Introduce polar coordinates (r, θ); along the solution curve they become functions of time $(r(t), \theta(t))$. Since $r^2\dot{\theta}$ is constant and not 0, *the sign of* $\dot{\theta}$ *is constant* along the curve. Thus θ is always increasing or always decreasing with time. Therefore r *is a function of* θ *along the curve.*

Let $u(t) = 1/r(t)$; then u is also a function of $\theta(t)$. Note that

$$u = -V.$$

We have a convenient formula for kinetic energy T.

Lemma

$$T = \frac{1}{2}\frac{h^2}{m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

Proof. From the formula for $\dot{x}$ in Section 4 and the definition of T we have

$$T = \tfrac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2].$$

Also,

$$\dot{r} = \frac{-1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{h}{m}\frac{du}{d\theta}$$

by the chain rule and the definitions of u and h; and also

$$r\dot{\theta} = \frac{h}{mr} = \frac{hu}{m}.$$

Substitution in the formula for T proves the lemma.

Now we find a differential equation relating u and θ along the solution curve. Observe that $T = E - V = E + u$. From the lemma we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{h^2}(E + u). \tag{1}$$

Differentiate both sides by θ, divide by $2\,du/d\theta$, and use $dE/d\theta = 0$ (conservation of energy). We obtain another equation

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{h^2}, \tag{2}$$

where m/h^2 is a constant.

We re-examine the meaning of just what we are doing and of (2). A particular orbit of the planar central force problem is considered, the force being gravitational. Along this orbit, the distance r from the origin (the source of the force) is a function of θ, as is $1/r = u$. We have shown that this function $u = u(\theta)$ satisfies (2), where h is the constant angular momentum and m is the mass.

The solution of (2) (as was seen in Section 1) is

$$u = \frac{m}{h^2} + C\cos(\theta + \theta_0), \tag{3}$$

where C and θ_0 are arbitrary constants.

To obtain a solution to (1), use (3) to compute $du/d\theta$ and $d^2u/d\theta^2$, substitute the resulting expression into (1) and solve for C. The result is

$$C = \pm\frac{1}{h^2}(2mh^2E + m^2)^{1/2}.$$