

# Preface

In this book we describe a new non-perturbative approach to the fermionic many-body problem, which can be considered as a generalization to arbitrary dimensions of the well-known bosonization technique for one-dimensional fermions. Our approach is based on the direct calculation of correlation functions of interacting Fermi systems with dominant forward scattering via functional integration and Hubbard-Stratonovich transformations – we do not attempt to generalize one-dimensional operator identities between the fermionic and bosonic field operators to higher dimensions. The fundamental ideas of higher-dimensional bosonization have first been formulated by A. Luther (1979) and by F. D. M. Haldane (1992). In this book we shall go beyond these ideas and develop a powerful and systematic method for bosonizing interacting fermions on a *curved* Fermi surface. We then apply our method to a number of problems of physical interest which are very difficult – and in some cases impossible – to solve by means of conventional diagrammatic perturbation theory.

The restriction to dominant forward scattering means that in real space the effective interaction between the fermions must be sufficiently long-range. Physical examples are the Coulomb interaction at high densities, or the effective current-current interaction mediated by transverse gauge fields. On the other hand, short-range interactions like the local Hubbard-interaction cannot be directly treated within our approach. It seems, however, that our method can be generalized to include (at least perturbatively) scattering processes with large momentum transfer. Although we shall restrict ourselves to *normal* Fermi systems, with our functional approach it should be straightforward to take also spontaneous symmetry breaking into account. We would like to encourage interested readers to contribute to the further development of our method. At the end of each chapter we have therefore mentioned open research problems, which might be solvable with the help of extensions of the methods developed in this book.

I would like to thank at this point everyone who – directly or indirectly – has helped me to complete this book. First of all, I am grateful to Kurt Schönhammer for numerous collaborations and discussions, for getting me interested in bosonization shortly after I had moved to Göttingen, and for giving me the freedom I needed to pursue my own ideas. The formal devel-

opment of the functional bosonization approach was partially carried out in collaboration with Kurt, and without him this approach would have never been formulated in this simplicity and clarity. More recently I have been collaborating with my friend Guillermo Castilla, on whom I could always count whenever I needed encouragement, advise, or help. We communicate mainly via E-mail, but my information exchange with Guillermo has been almost as intense as during our common time as graduate students at UCLA.

I am also grateful to Sudip Chakravarty and Konstantin Efetov for being my teachers. Under Sudip's guidance I have learnt to do independent research. He has taught me to distinguish interesting physics from empty mathematics, and his very intuitive way of thinking about physical problems has strongly influenced my personal style of choosing and solving my own research projects. I have enjoyed very much being a postdoc in Konstantin Efetov's international and very active group at the *Max-Planck-Institut für Festkörperforschung* at Stuttgart. During this time I could broaden my horizon and become familiar with the physics of disordered Fermi systems. I have greatly profited from Konstantin's profound knowledge in this field.

I would like to thank Peter Wölfle for comments on the manuscript, and for pointing out some references related to gauge fields. In one way or the other, I have also profited from discussions and collaborations with Lorenz Bartosch, Jim "Claude" Cochran, Fabian "Fabman" Essler, Jens Fricke, Lev Gehlhoff, Ralf Hannappel, Joachim Hermisson, Jens Kappey, Stefan Kettmann, Volker Meden, Walter Metzner, Jacob Morris, Ben Sauer, Peter Scharf, Axel Völker, and Roland Zeyher.

Although I sometimes tend to ignore it, I know very well that there are more important things in life than physics. This book is dedicated to my girlfriend Cornelia Buhrke for helping me to keep in touch with the real world during the nearly two years of writing, and for much more...

Göttingen, December 1996

*Peter Kopietz*

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## Development of the formalism





# 1. Introduction

*...in which we try to explain why we have written this book.*

## 1.1 Perturbation theory and quasi-particles

*Perturbation theory for the single-particle Green's function of an interacting Fermi system usually works as long as the quasi-particle picture is valid.*

The long-wavelength and low-energy behavior of the single-particle Green's function  $G(\mathbf{k}, \omega)$  of an interacting many-body system is directly related to the nature of its ground state and low lying excited states [1.1–1.6]. Because the qualitative features of the low-energy spectrum of a many-body Hamiltonian are usually determined by certain universal parameters such as dimensionality, symmetries, and conservation laws [1.7], the infrared behavior of the single-particle Green's function can be used to classify interacting many-body systems. Moreover, if  $G(\mathbf{k}, \omega)$  is known for all wave-vectors  $\mathbf{k}$  and frequencies  $\omega$ , one can in principle calculate all thermodynamic properties of the system [1.6]. Unfortunately, in almost all physically interesting cases it is impossible to calculate the Green's function exactly, so that one has to resort to approximate methods. The most naive approach would be the direct expansion of  $G(\mathbf{k}, \omega)$  in powers of the interaction. It is well known, however, that even for small interactions such an expansion is not valid for all wave-vectors and frequencies, because  $G(\mathbf{k}, \omega)$  usually has poles or other singularities, in the vicinity of which a power series expansion of  $G(\mathbf{k}, \omega)$  is not possible. In many cases this problem can be avoided if one introduces the irreducible self-energy  $\Sigma(\mathbf{k}, \omega)$  via the Dyson equation,

$$[G(\mathbf{k}, \omega)]^{-1} = [G_0(\mathbf{k}, \omega)]^{-1} - \Sigma(\mathbf{k}, \omega) \quad , \quad (1.1)$$

and calculates  $\Sigma(\mathbf{k}, \omega)$  instead of  $G(\mathbf{k}, \omega)$  in powers of the interaction. Here  $G_0(\mathbf{k}, \omega)$  is the Green's function of a suitably defined non-interacting system, which can be calculated exactly. It is important to stress that the Dyson equation does not simply express one unknown quantity  $G(\mathbf{k}, \omega)$  in terms of another unknown  $\Sigma(\mathbf{k}, \omega)$ , but tells us that the *inverse* Green's function should be expanded in powers of the interaction.

In so-called *Landau Fermi liquids* the above perturbative approach can indeed be used to calculate the Green's function. Of course, for strong interactions infinite orders in perturbation theory have to be summed, but the integrals generated in the perturbative expansion are free of divergencies and lead to a finite expression for the self-energy. The theory of Fermi liquids was advanced by Landau [1.8] in 1956 as a phenomenological theory to describe the static and dynamic properties of a large class of interacting fermions [1.9]. The most important physical realization of a Fermi liquid are electrons in clean three-dimensional metals, but also liquid  ${}^3\text{He}$  is a Fermi liquid [1.10]. Simultaneously with Landau's pioneering ideas the powerful machinery of quantum field theory was developed and applied to condensed matter systems [1.1–1.5], and a few years later his phenomenological theory was put on a solid theoretical basis [1.9]. The retarded single-particle Green's function<sup>1</sup> of a Fermi liquid is for wave-vectors  $\mathbf{k}$  in the vicinity of the Fermi surface and small frequencies  $\omega$  to a good approximation given by

$$G(\mathbf{k}, \omega + i0^+) \approx \frac{Z_{\mathbf{k}}}{\omega - \tilde{\xi}_{\mathbf{k}} + i\gamma_{\mathbf{k}}} \quad , \quad (1.2)$$

where the number  $Z_{\mathbf{k}}$  is the so-called *quasi-particle residue*, and the energy  $\tilde{\xi}_{\mathbf{k}}$  is the single-particle excitation energy. Because by definition Landau Fermi liquids are metals, the excitation energy  $\tilde{\xi}_{\mathbf{k}}$  must be gapless. This means that there exists a surface in  $\mathbf{k}$ -space where  $\tilde{\xi}_{\mathbf{k}} = 0$ . In a Fermi liquid this equation can be used to *define* the Fermi surface. The positive energy  $\gamma_{\mathbf{k}}$  in Eq.(1.2) can be identified with the quasi-particle damping, and is assumed to vanish faster than  $\tilde{\xi}_{\mathbf{k}}$  when the wave-vector  $\mathbf{k}$  approaches the Fermi surface. Note that in the complex  $\omega$ -plane  $G(\mathbf{k}, \omega + i0^+)$  has a simple pole at  $\omega = \tilde{\xi}_{\mathbf{k}} - i\gamma_{\mathbf{k}}$  with residue  $Z_{\mathbf{k}}$ . Obviously, the Green's function of *non-interacting* fermions can be obtained as a special case of Eq.(1.2), namely by setting  $Z_{\mathbf{k}} = 1$ ,  $\gamma_{\mathbf{k}} = 0^+$ , and identifying  $\tilde{\xi}_{\mathbf{k}}$  with the non-interacting energy dispersion measured relative to the chemical potential. Then the pole at  $\omega = \tilde{\xi}_{\mathbf{k}} - i0^+$  with unit residue is a consequence of the undamped propagation of a particle with energy dispersion  $\tilde{\xi}_{\mathbf{k}}$  through the system. The corresponding pole in the Green's function of an interacting Fermi liquid is associated with a so-called *quasi-particle*. The important point is that in the vicinity of the quasi-particle pole the Green's function of a Fermi liquid has *qualitatively* the same structure as the Green's function of free fermions. In renormalization group language, the interacting Fermi liquid and the free Fermi gas correspond to the same fixed point in the infinite-dimensional parameter space spanned by all possible scattering processes [1.11, 1.12]. As explained in detail in Chap. 2, in a Landau Fermi liquid the quantities  $Z_{\mathbf{k}}$ ,  $\tilde{\xi}_{\mathbf{k}}$  and  $\gamma_{\mathbf{k}}$  can be calculated from the derivatives of the self-energy  $\Sigma(\mathbf{k}, \omega)$ .

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<sup>1</sup> We denote the Fourier transform of the *time-ordered* Green's function at wave-vector  $\mathbf{k}$  and frequency  $\omega$  by  $G(\mathbf{k}, \omega)$ . The corresponding retarded Green's function will be denoted by  $G(\mathbf{k}, \omega + i0^+)$ , and the advanced one by  $G(\mathbf{k}, \omega - i0^+)$ .

In some cases, however, the application of the standard machinery of many-body theory leads to divergent integrals in the perturbative expansion of  $\Sigma(\mathbf{k}, \omega)$ . The breakdown of perturbation theory is a manifestation of the fact that the interacting Green's function is not any more related in a simple way to the non-interacting one. In this case the system cannot be a Fermi liquid. A well known example are electrons in one spatial dimension with regular interactions, which under quite general conditions show *Luttinger liquid* behavior [1.13–1.15]. In contrast to a Fermi liquid, the Green's function of a Luttinger liquid does not have simple poles in the complex frequency plane, but exhibits only branch cut singularities involving non-universal power laws<sup>2</sup>. As a consequence, in a Luttinger liquid  $[G(\mathbf{k}, \omega)]^{-1}$  cannot be calculated by simple perturbation theory around  $[G_0(\mathbf{k}, \omega)]^{-1}$ . Hence, non-perturbative methods are necessary to calculate the Green's function of interacting fermions in  $d = 1$  dimension. Besides the Bethe ansatz [1.16] and renormalization group methods [1.13], the bosonization approach has been applied to one-dimensional Fermi systems with great success [1.13–1.15]. Over the past 30 years numerous interesting results have been obtained with this non-perturbative method. The so-called Tomonaga-Luttinger model is a paradigm for an exactly solvable non-trivial many-body system which exhibits all the characteristic Luttinger liquid properties, such as the absence of a quasi-particle peak in the single-particle Green's function, anomalous scaling, and spin-charge separation [1.17–1.19]. Even now interesting new results on the Tomonaga-Luttinger model are reported in the literature [1.20, 1.21]. For an up-to-date overview and extensive references on bosonization in  $d = 1$  we would like to refer the reader to the recent reprint volume by M. Stone [1.22]. The central topic of this book is the generalization of the bosonization approach to arbitrary dimensions.

## 1.2 A brief history of bosonization in $d > 1$

*We apologize in advance if we should have forgotten someone. Maybe some Russians have bosonized higher-dimensional Fermi systems long time ago, and we just don't know about their work . . .*

The discovery of the high-temperature superconductors and Anderson and co-workers suggestion [1.23, 1.24] that the normal-state properties of these materials are a manifestation of non-Fermi liquid behavior in dimensions  $d > 1$  has revived the interest to develop non-perturbative methods for analyzing interacting fermions in  $d > 1$ . Note, however, that for regular interactions in  $d > 1$  perturbation theory is consistent in the sense that within the framework

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<sup>2</sup> In Chap. 6.3 we shall discuss the behavior of the Green's function of Luttinger liquids in some detail.

of perturbation theory itself there is no signal for its breakdown [1.25, 1.26]. Nevertheless, consistency of perturbation theory does not imply that the perturbative result must be correct. It is therefore highly desirable to analyze interacting Fermi systems by means of a non-perturbative approach which does not assume *a priori* that the system is a Fermi liquid. The recently developed higher-dimensional generalization of bosonization seems to be the most promising analytical method which satisfies this criterion in  $d > 1$ .

In one dimension bosonization is based on the observation that, after proper rescaling, the operators describing density fluctuations obey canonical bosonic commutation relations [1.13–1.15]. But also in  $d = 3$  density fluctuations in an interacting Fermi system behave in many respects like bosonic degrees of freedom [1.27, 1.28]. The first serious attempt to formalize this observation and exploit it to develop a generalization of the one-dimensional bosonization approach to arbitrary dimensions was due to Luther [1.29]. However, Luther's pioneering work has not received much attention until Haldane [1.30] added the grain of salt that was necessary to turn higher-dimensional bosonization into a practically useful non-perturbative approach to the fermionic many-body problem. Haldane's crucial insight was that the degrees of freedom in the vicinity of the Fermi surface should be subdivided into boxes of *finite cross section*, such that *the motion of particle-hole pairs can be described without taking momentum-transfer between different boxes into account*. In Luther's formulation only the motion normal to the Fermi surface can be described in such a simple way. The first applications of Haldane's bosonization ideas to problems of physical interest were given by Houghton, Marston and Kwon [1.31], and independently by Castro Neto and Fradkin [1.32]. These approaches follow closely the usual bosonization procedure in one-dimensional systems, and are based on higher-dimensional generalizations of the Kac-Moody algebra that is *approximately* satisfied by charge and spin current operators. Just like in  $d = 1$ , it is possible to map with this method the fermionic many-body Hamiltonian onto an effective non-interacting bosonic Hamiltonian. The potential of these operator bosonization approaches is certainly not yet exhausted [1.33, 1.34]. However, unlike recent claims in the literature [1.34], bosonization in  $d > 1$  is *not exact*. For example, scattering processes that transfer momentum between different boxes on the Fermi surface and non-linear terms in the energy dispersion definitely give rise to corrections to the free-boson approximation for the Hamiltonian. The problem of calculating these corrections within the conventional operator approach seems to be very difficult and so far has not been solved.

In the present book we shall develop an alternative generalization of the bosonization approach to arbitrary dimensions, which is based on functional integration and Hubbard-Stratonovich transformations. In this way we avoid the algebraic considerations of commutation relations which form the basis of the operator bosonization approaches [1.31, 1.32]. The functional integral formulation of higher-dimensional bosonization has been developed by the

author in collaboration with Kurt Schönhammer [1.35] during spring 1994. Since then we have considerably refined this method [1.36–1.38] and applied it to various problems of physical interest. A coherent and detailed presentation of these results will be given in this book. A similar functional bosonization method, which emphasizes more the mathematical aspects of bosonization, has been developed independently by Fröhlich and collaborators [1.39, 1.40]. In the context of the one-dimensional Tomonaga-Luttinger model the functional bosonization technique has first been discussed by Fogedby [1.41], and later by Lee and Chen [1.42].

Compared with the more conventional operator bosonization [1.31–1.34], the functional bosonization approach has several advantages. The most important advantage is that within our functional integral approach it is possible to *handle the non-linear terms in the energy dispersion* (and hence in  $d > 1$  the *curvature* of the Fermi surface). Note that the linearization of the energy dispersion close to the Fermi surface is one of the crucial (and a priori uncontrolled) approximations of conventional bosonization; even in  $d = 1$  it is very difficult to calculate systematically the corrections due to the non-linear terms in the expansion of the dispersion relation close to the Fermi surface [1.15, 1.43]. A practically useful method for doing this will be developed in this book. In Chap. 4 we shall explicitly calculate the leading correction to the free bosonized Hamiltonian and the density-density correlation function. Moreover, in Chap. 5.2 we shall show how the bosonization result for the single-particle Green's function for fermions with linearized energy dispersion is modified by the quadratic term in the expansion of the energy dispersion close to the Fermi surface. In this way the approximations inherent in higher-dimensional bosonization become very transparent.

Another advantage of the functional integral formulation of higher-dimensional bosonization is that it can be applied in a straightforward way to physical problems where non-locality and retardation are essential. It is well-known [1.44] that these important many-body effects can be described in the most simple and general way via functional integrals and effective actions. In fact, the complicated effective dynamics of a quantum mechanical system that is coupled to another subsystem can sometimes only be described by means of a non-local effective action, and not by a Hamiltonian [1.45]. For example, the effective retarded interaction between electrons that is mediated via phonons or photons cannot be represented in terms of a conventional Hamiltonian. It is therefore advantageous to use functional integrals and the concept of an effective action as a basis to generalize the bosonization approach to dimensions larger than one.

Alternative formulations of higher-dimensional bosonization have also been proposed by Schmelzer and Bishop [1.46], by Khveshchenko and collaborators [1.47, 1.48], and by Li [1.49]. In particular, Khveshchenko [1.48] has also developed a formal method to include the curvature of the Fermi surface into higher-dimensional bosonization. However, so far his method has not

been proven to be useful in practice. We shall not further discuss the above works in this book, because we believe that our functional bosonization technique leads to a more transparent and practically more useful approach to the bosonization problem in arbitrary dimensions. Finally, it should be mentioned that recently Castellani, Di Castro and Metzner [1.50–1.52] have proposed another non-perturbative approach to the fermionic many-body problem in  $d > 1$ . Their method is based on Ward identities and sums exactly the same infinite number of Feynman diagrams in the perturbation series as higher-dimensional bosonization with linearized energy dispersion. We shall derive the precise relation between the Ward identity approach and bosonization in Chap. 5.1.4.

### 1.3 The scope of this book

We have subdivided this book into two parts. Part I comprises the first five chapters and is devoted to the formal development of the functional bosonization approach. We begin by reminding the reader in Chap. 2 of some basic facts about interacting fermions. We also describe in some detail various ways of subdividing the momentum space in the vicinity of the Fermi surface into sectors. These geometric constructions are the key to the generalization of the bosonization approach to arbitrary dimensions. In Chap. 3 we introduce two Hubbard-Stratonovich transformations which directly lead to the bosonization result for the single-particle Green's function and the boson representation of the Hamiltonian. The explicit calculation of the bosonic Hamiltonian is presented in Chap. 4, where we also show that the problem of bosonizing the Hamiltonian is essentially equivalent with the problem of calculating the density-density correlation function. We also show that the non-Gaussian terms in the bosonic Hamiltonian are closely related to the local field corrections to the random-phase approximation. Chapter 5 is devoted to the calculation of the single-particle Green's function. This is the most important chapter of this book, because here we describe in detail our non-perturbative method for including the non-linear terms in the expansion of the energy dispersion for wave-vectors close to the Fermi surface into the bosonization procedure. Note that in  $d > 1$  the local *curvature* of the Fermi surface can only be described if the quadratic term in the energy dispersion is retained. Our method is based on a generalization of the Schwinger ansatz for the Green's function in a given external field, an imaginary-time eikonal expansion, and diagrammatic techniques borrowed from the theory of disordered systems.

In Part II we shall use our formalism to calculate and classify the long-wavelength and low-energy behavior of a number of normal fermionic quantum liquids. In most cases we shall concentrate on parameter regimes where conventional perturbation theory is not applicable. In particular, we discuss fermions with singular density-density interactions (Chap. 6), quasi-

one-dimensional metals (Chap. 7), electron-phonon interactions (Chap. 8), electrons in a dynamic random medium (Chap. 9), and fermions that are coupled to transverse gauge fields (Chap. 10.). Finally, in the Appendix we summarize some useful results on screening and collective modes in arbitrary dimensions.

Because the method described in this book is rather new, much remains to be done to establish higher-dimensional bosonization as a generally accepted, practically useful non-perturbative tool for studying strongly correlated Fermi systems. We would like to encourage all readers to actively participate in the process of further developing this method. For this purpose we have given at the end of each chapter a brief summary of the main results, together with a list of open problems and possible directions for further research.

## 1.4 Notations and assumptions

Let us briefly summarize the conventions that will be used throughout this work. We shall measure temperature  $T$  and frequencies  $\omega$  in units of energy, which amounts to formally setting the Boltzmann constant  $k_B$  and Planck's constant  $\hbar$  equal to unity. Note that in these units it is not necessary to distinguish between wave-vectors and momenta. The charge of the electron will be denoted by  $-e$ , and the fine structure constant is  $\alpha = \frac{e^2}{c} \approx \frac{1}{137}$ . The velocity of light  $c$  will not be set equal to unity, because in our discussion of transverse gauge fields in Chap. 10 it is useful to explicitly see the ratio  $v_F/c$ , where  $v_F$  is the Fermi velocity. The inverse temperature will be denoted by  $\beta = 1/T$ , and the volume of the system by  $V$ . Although at intermediate steps the volume of space-time  $V\beta$  will be held finite, we are eventually interested in the limits of infinite volume ( $V \rightarrow \infty$ ) and zero temperature ( $\beta \rightarrow \infty$ ). As pointed out by Kohn, Luttinger, and Ward [1.53], in case of ambiguities the limit  $V \rightarrow \infty$  should be taken before the limit  $\beta \rightarrow \infty$ . However, we shall ignore the subtleties associated with the infinite volume limit that have recently been discussed by Metzner and Castellani [1.54]. Although we are interested in the zero-temperature limit, we shall use the Matsubara formalism and work at intermediate steps at finite temperatures. In this way we also eliminate possible unphysical ‘‘anomalous’’ terms [1.53] which sometimes appear in a zero-temperature formalism, but are avoided if the Matsubara sums are performed at finite temperature and the  $T \rightarrow 0$  limit is carefully taken afterwards.

We shall denote bosonic Matsubara frequencies by  $\omega_m = 2\pi mT$ ,  $m = 0, \pm 1, \pm 2, \dots$ , and put an extra tilde over fermionic ones,  $\tilde{\omega}_n = 2\pi[n + \frac{1}{2}]T$ ,  $n = 0, \pm 1, \pm 2, \dots$ . To simplify the notation, we introduce composite labels for wave-vectors and Matsubara frequencies:  $k \equiv [\mathbf{k}, i\tilde{\omega}_n]$ ,  $q \equiv [\mathbf{q}, i\omega_m]$ , and  $\tilde{q} \equiv [\mathbf{q}, i\tilde{\omega}_n]$ . Note that the label  $q$  is associated with bosonic frequencies, whereas  $k$  and  $\tilde{q}$  involve fermionic frequencies.





## 2. Fermions and the Fermi surface

We summarize some basic facts about interacting fermions and introduce notations that will be used throughout this book. We also describe Haldane's way of partitioning the Fermi surface into patches and generalize it such that the curvature of the Fermi surface can be taken into account.

### 2.1 The generic many-body Hamiltonian

We first introduce the many-body Hamiltonian for interacting fermions and point out some subtleties associated with ultraviolet cutoffs.

The starting point of conventional many-body theory is a second-quantized Hamiltonian of the form

$$\hat{H}_{\text{mat}} = \hat{H}_0 + \hat{H}_{\text{int}} \quad , \quad (2.1)$$

$$\hat{H}_0 = \sum_{\mathbf{k}} \sum_{\sigma} \epsilon_{\mathbf{k}} \hat{\psi}_{\mathbf{k}\sigma}^{\dagger} \hat{\psi}_{\mathbf{k}\sigma} \quad , \quad (2.2)$$

$$\hat{H}_{\text{int}} = \frac{1}{2V} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} \sum_{\sigma\sigma'} f_{\mathbf{q}}^{\mathbf{k}\sigma\mathbf{k}'\sigma'} \hat{\psi}_{\mathbf{k}+\mathbf{q}\sigma}^{\dagger} \hat{\psi}_{\mathbf{k}'-\mathbf{q}\sigma'}^{\dagger} \hat{\psi}_{\mathbf{k}'\sigma'} \hat{\psi}_{\mathbf{k}\sigma} \quad , \quad (2.3)$$

where  $\hat{\psi}_{\mathbf{k}\sigma}$  and  $\hat{\psi}_{\mathbf{k}\sigma}^{\dagger}$  are canonical annihilation and creation operators for fermions with wave-vector  $\mathbf{k}$  and spin  $\sigma$ , which satisfy the anti-commutation relations

$$[\hat{\psi}_{\mathbf{k}\sigma}, \hat{\psi}_{\mathbf{k}'\sigma'}^{\dagger}]_{+} = \hat{\psi}_{\mathbf{k}\sigma} \hat{\psi}_{\mathbf{k}'\sigma'}^{\dagger} + \hat{\psi}_{\mathbf{k}'\sigma'}^{\dagger} \hat{\psi}_{\mathbf{k}\sigma} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'} \quad . \quad (2.4)$$

The quantities  $f_{\mathbf{q}}^{\mathbf{k}\sigma\mathbf{k}'\sigma'}$  are the so-called *Landau interaction parameters*, describing the scattering of two particles from initial states with quantum numbers  $(\mathbf{k}, \sigma)$  and  $(\mathbf{k}', \sigma')$  into final states with quantum numbers  $(\mathbf{k} + \mathbf{q}, \sigma)$  and  $(\mathbf{k}' - \mathbf{q}, \sigma')$ . This process can be represented graphically by the Feynman diagram shown in Fig. 2.1. Quantum many-body theory is usually formulated in the grand canonical ensemble, where the relevant combination is  $\hat{H}_{\text{mat}} - \mu \hat{N}$ . Here  $\hat{N} = \sum_{\mathbf{k}} \sum_{\sigma} \hat{\psi}_{\mathbf{k}\sigma}^{\dagger} \hat{\psi}_{\mathbf{k}\sigma}$  is the particle number operator, and  $\mu$  is the chemical potential. Thus, the energy dispersion  $\epsilon_{\mathbf{k}}$  appears exclusively in the combination

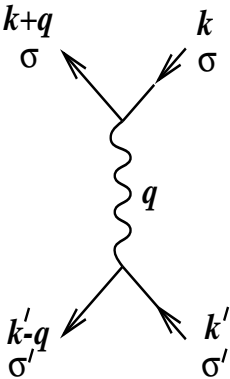


Fig. 2.1. Feynman diagram representing the interaction  $f_q^{k\sigma k'\sigma'}$  in Eq.(2.3).

$$\xi_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu \quad . \quad (2.5)$$

The value of  $\mu$  at zero temperature is also called the Fermi energy  $E_F$ . Although in most physical applications we are interested in three dimensions, it is very useful and instructive to formulate the theory in arbitrary dimension  $d$ . Then the equation

$$\xi_{\mathbf{k}} = 0 \quad (2.6)$$

defines a  $d - 1$ -dimensional surface in momentum space, the *non-interacting* Fermi surface. The precise definition of the *interacting* Fermi surface will be given in Sect. 2.2. Note that in  $d = 1$  the non-interacting Fermi surface consists of two distinct points  $\pm k_F$ , where  $k_F$  is the Fermi wave-vector. In higher dimensions the Fermi surface is a  $d - 1$ -dimensional manifold, the topology of which depends on the form of  $\xi_{\mathbf{k}}$ . There is actually a subtle point hidden in the above definition: although the energy  $\epsilon_{\mathbf{k}}$  is a parameter of the non-interacting Hamiltonian  $\hat{H}_0$ , the chemical potential  $\mu$  is by definition *the exact chemical potential of the interacting many-body system*. Of course, the precise value of  $\mu$  remains unknown unless we can solve the many-body problem, but fortunately it is not necessary to know  $\mu$  in order to calculate physical correlation functions. By defining  $\mu$  to be the chemical potential of the interacting many-body system, one implicitly adds a suitable counter-term to the bare chemical potential which eliminates, order by order in perturbation theory, all terms which would otherwise contribute to  $\Sigma(\mathbf{k}, 0)$  for wave-vectors  $\mathbf{k}$  on the Fermi surface. In particular, all Feynman diagrams of the Hartree type are cancelled by the counter-term. Such a procedure is familiar from perturbative quantum field theory [2.1]. The consistency for such a construction is by no means obvious, and has recently been questioned by Anderson [2.2]. For a thorough discussion and partial solution of this problem see [2.3].

It should be emphasized that Eqs.(2.1)–(2.3) can be interpreted in three distinct ways, which can be classified according to the effective ultraviolet cutoff for the wave-vector sums.

(a) *Homogeneous electron gas.* First of all, we may define  $\hat{H}_{\text{mat}}$  to be the Hamiltonian of the homogeneous electron gas in  $d$  dimensions. For example the Coulomb-interaction in  $d = 3$  dimensions corresponds to  $\epsilon_{\mathbf{k}} = \mathbf{k}^2/(2m)$  and  $f_{\mathbf{q}}^{k\sigma k'\sigma'} = 4\pi e^2/\mathbf{q}^2$ , where  $m$  is the mass of the electrons. In this case there is no intrinsic short-distance cutoff for the wave-vector sums.

(b) *Relevant band of a lattice model.* Because in realistic materials the electrons feel the periodic potential due to the ions, the allowed energies in the absence of interactions are subdivided into energy bands, and the interaction has interband matrix elements. But if there exists only a single band in the vicinity of the Fermi surface, then it is allowed to ignore all other bands as long as one is interested in energy scales small compared with the interband gap. In this case the Hamiltonian defined in Eqs.(2.1)–(2.3) should be considered as the *effective Hamiltonian* for the band in the vicinity of the Fermi energy. In this model the wave-vector sums have a cutoff of the order of  $2\pi/a$ , where  $a$  is the distance between the ions. The energy dispersion  $\epsilon_{\mathbf{k}}$  in Eq.(2.2) incorporates then by definition the effects of the underlying lattice, which in general leads also to a renormalization of the effective mass of the electrons.

(c) *Effective Hamiltonian for degrees of freedom close to the Fermi surface.* Finally, we may define  $\hat{H}_{\text{mat}}$  to be the effective Hamiltonian for the low-energy degrees of freedom in the vicinity of the Fermi surface, assuming that all degrees of freedom outside a thin shell with radial thickness  $\lambda \ll k_{\text{F}}$  have been integrated out via functional integration and renormalization group methods [2.12]. Of course, the operation of integrating out the high-energy degrees of freedom will also generate three-body and higher order interactions, which are ignored in Eqs.(2.1)–(2.3). The quantities  $\epsilon_{\mathbf{k}}$  and  $f_{\mathbf{q}}^{k\sigma k'\sigma'}$  should then be considered as effective parameters, which take the finite renormalizations due to the high-energy degrees of freedom into account. In this picture the  $\mathbf{k}$ - and  $\mathbf{k}'$ -sums in Eqs.(2.2) and (2.3) are confined to a thin shell of thickness  $\lambda$  around the Fermi surface, while the  $\mathbf{q}$ -sum in Eq.(2.3) is restricted to the regime  $|\mathbf{q}| \leq \lambda$ .

All three interpretations of the many-body Hamiltonian (2.1)–(2.3) are useful. First of all, the model (a) has the advantage that it contains no free parameters, so that it can be the starting point of a first principles microscopic calculation. The model (b) is more realistic, although the effects of the underlying lattice are only included on a phenomenological level. Finally, the model (c) has the advantage that it contains explicitly only the low-energy degrees of freedom close to the Fermi surface, so that, to a first approximation, we may locally linearize the energy dispersion at the Fermi surface. Evidently the model (c) cannot be used for the calculation of the precise numerical value of physically measurable quantities that depend on fluctuations on all length scales. Furthermore, the integration over the degrees of freedom far away from the Fermi surface usually cannot be explicitly carried out.

## 2.2 The single-particle Green's function

We define the single-particle Green's function and the Fermi surface of an interacting Fermi system. We then discuss in some detail the low-energy behavior of the Green's function in a Landau Fermi liquid.

Because in the rest of this book the spin degree of freedom will not play any role, we shall from now on simply ignore the spin index. Formally, the spin is easily taken into account by defining  $\mathbf{k}$  and  $\mathbf{k}'$  to be collective labels for wave-vector and spin. For practical calculations we prefer to work with the Matsubara formalism, because in this way we avoid the problem of regularizing formally divergent integrals by means of pole prescriptions, which arises in the real time zero-temperature formulation of quantum many-body theory. Furthermore, the Matsubara Green's function can be represented as an imaginary time functional integral [2.4–2.7], so that the entire many-body problem can be reformulated in the language of path integrals. In this work we shall make extensive use of this modern approach to the many-body problem.

### 2.2.1 Definition of the Green's function

The single particle Matsubara Green's function  $G(k)$  of an interacting Fermi system is defined by

$$G(k) \equiv G(\mathbf{k}, i\tilde{\omega}_n) = -\frac{1}{\beta} \int_0^\beta d\tau \int_0^\beta d\tau' e^{-i\tilde{\omega}_n(\tau-\tau')} \langle \mathcal{T} [\hat{\psi}_{\mathbf{k}}(\tau) \hat{\psi}_{\mathbf{k}}^\dagger(\tau')] \rangle , \quad (2.7)$$

where for fermions the time-ordering operator  $\mathcal{T}$  in imaginary time is defined by

$$\begin{aligned} \mathcal{T} [\hat{\psi}_{\mathbf{k}}(\tau) \hat{\psi}_{\mathbf{k}}^\dagger(\tau')] &= \Theta(\tau - \tau' - 0^+) \hat{\psi}_{\mathbf{k}}(\tau) \hat{\psi}_{\mathbf{k}}^\dagger(\tau') \\ &\quad - \Theta(\tau' - \tau + 0^+) \hat{\psi}_{\mathbf{k}}^\dagger(\tau') \hat{\psi}_{\mathbf{k}}(\tau) , \end{aligned} \quad (2.8)$$

and the average in Eq.(2.7) denotes grand canonical thermal average with respect to all degrees of freedom in the system. For any operator  $\hat{O}$  the time evolution in imaginary time is defined by

$$\hat{O}(\tau) = e^{\tau(\hat{H}_{\text{mat}} - \mu\hat{N})} \hat{O} e^{-\tau(\hat{H}_{\text{mat}} - \mu\hat{N})} , \quad (2.9)$$

where  $\hat{H}_{\text{mat}}$  is given in Eqs.(2.1)–(2.3). The Matsubara Green's function of a system of non-interacting fermions with Hamiltonian  $\hat{H}_0$  (see Eq.(2.2)) is given by

$$G_0(k) = \frac{1}{i\tilde{\omega}_n - \xi_{\mathbf{k}}} , \quad (2.10)$$