CONTINUOUS SYMMETRIES, LIE ALGEBRAS, DIFFERENTIAL EQUATIONS AND COMPUTER ALGEBRA

2nd Edition

Willi-Hans Steeb

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Preface

The purpose of this book is to provide a comprehensive introduction to the application of continuous symmetries and their Lie algebras to ordinary and partial differential equations. The study of symmetries of differential equations provides important information about the behaviour of differential equations. The symmetries can be used to find exact solutions. They can be applied to verify and develop numerical schemes. One can also obtain conservation laws of a given differential equation with the help of the continuous symmetries. Gauge theory is also based on the continuous symmetries of certain relativistic field equations.

Apart from the standard techniques in the study of continuous symmetries, the book includes: the Painlevé test and symmetries, invertible point transformation and symmetries, Lie algebra valued differential forms, gauge theory, Yang-Mills theory and chaos, self-dual Yang-Mills equation and soliton equations, Bäcklund transformation, Lax representation, Bose operators and symmetries, Hirota technique, Sato's theory, discrete systems and invariants and string theory.

Each chapter includes computer algebra applications. Examples are the finding of the determining equation for the Lie symmetries, finding the curvature for a given metric tensor field and calculating the Killing vector fields for a metric tensor field. Each chapter also includes exercises.

The book is suitable for use by students and research workers whose main interest lies in finding solutions of differential equations. It therefore caters for readers primarily interested in applied mathematics and physics rather than pure mathematics. The book provides an application focused text that is self-contained. A large number of worked examples have been included in the text to help the readers working independently of a teacher. The advance of algebraic computation has made it possible to write programs for the tedious calculations in this research field. Thus the last chapter gives a survey on computer algebra packages. Each chapter also includes useful SymbolicC++ programs.

End of proofs are indicated by ♠. End of examples are indicated by ♣.

I wish to express my gratitude to Yorick Hardy for discussion on this research field and the support for SymbolicC++.

Any useful suggestions and comments are welcome.

The book covers the course on Lie groups and Lie algebras provided by the International School for Scientific Computing. If you are interest in such a course please contact the author.

The header files for SymbolicC++ and example programs can be downloaded from the home page of the author:

`http://issc.uj.ac.za`

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Contents

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Notation

$\emptyset$	empty set
$\mathbf{N}$	natural numbers
$\mathbf{Z}$	integers
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{R}^+$	nonnegative real numbers
$\mathbf{C}$	complex numbers
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{C}^n$	n-dimensional complex linear space
G	group
M, N	manifolds
$f : M \to N$	mapping between manifolds
TM	tangent bundle
T^*M	cotangent bundle
i	$:= \sqrt{-1}$
z	complex number
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\mathbf{x} \in \mathbf{R}^n$	element $\mathbf{x}$ of $\mathbf{R}^n$
$A \subset B$	subset A of set B
$A \cap B$	the intersection of the sets A and B
$A \cup B$	the union of the sets A and B
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
u	dependent variable
t	independent variable (time variable)
x	independent variable (space variable)
$\mathbf{x}^T = (x_1, x_2, \ldots, x_m)$	vector of independent variables, ^{T} transpose
$\mathbf{u}^T = (u_1, u_2, \ldots, u_n)$	vector of dependent variables, ^{T} transpose
$\| . \|$	norm
$\mathbf{x} \cdot \mathbf{y}$	scalar product (inner product)

$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
$\otimes$	Kronecker product, tensor product
det	determinant of a square matrix
tr	trace of a square matrix
I	unit matrix
$[\, , \,]$	commutator
$[\, , \,]_+$	anticommutator
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
$\wedge$	Grassmann product (exterior product)
d	exterior derivative
$*$	Hodge star operator
H	Hamilton function
L	Lagrange function
$\mathcal{L}$	Lagrange density
b	Bose annihilation operators
$b^\dagger$	Bose creation operator
c	Fermi annihilation operator
$c^\dagger$	Fermi creation operator

Chapter 1

Introduction

Sophus Lie (1842–1899) and Felix Klein (1849–1925) studied mathematical systems from the perspective of those transformation groups which left the systems invariant. Klein, in his famous "Erlanger" program, pursued the role of finite groups in the studies of regular bodies and the theory of algebraic equations, while Lie developed his notion of continuous transformation groups and their role in the theory of differential equations. Today the theory of continuous groups is a fundamental tool in such diverse areas as analysis, differential geometry, number theory, atomic structure and high-energy physics. In this book we deal with Lie's theorems and extensions thereof, namely its applications to the theory of differential equations.

It is well known that many, if not all, of the fundamental equations of physics are nonlinear and that linearity is achieved as an approximation. One of the important developments in applied mathematics and theoretical physics over the recent years is that many nonlinear equations, and hence many nonlinear phenomena, can be treated as they are, without approximations, and be solved by essentially linear techniques.

One of the standard techniques for solving linear partial differential equations is the Fourier transform. During the past 35 years it was shown that a class of physically interesting nonlinear partial differential equations can be solved by a nonlinear extension of the Fourier technique, namely the inverse scattering transform. This reduces the solution of the Cauchy problem to a series of linear steps. This method, originally applied to the Korteweg-de Vries equation, is now known to be applicable to a large class of nonlinear evolution equations in one space and one time variable, to quite a few equations in $2+1$ dimensions and also to some equations in higher dimensions.

Continuous group theory, Lie algebras and differential geometry play an important role in the understanding of the structure of nonlinear partial differential equations, in particular for generating integrable equations, finding Lax pairs, recursion operators, Bäcklund transformations and finding exact analytic solutions.

Most nonlinear equations are not integrable and cannot be treated via the inverse scattering transform, nor its generalizations. They can of course be treated by numerical methods, which are the most common procedures. Interesting qualitative and quantitative features are however often missed in this manner and it is of great value to be able to obtain, at least, particular exact analytic solutions of nonintegrable equations. Here group theory and Lie algebras play an important role. Indeed, Lie group theory was originally created as a tool for solving ordinary and partial differential equations, be they linear or nonlinear.

New developments have also occurred in this area. Some of them have their origins in computer science. The advent of algebraic computing and the use of such computer languages for symbolic computations such as SymbolicC++, REDUCE, MACSYMA, AXIOM, MAPLE, MATHEMATICA, MuPAD etc., have made it possible (in principle) to write computer programs that construct the Lie algebra of the symmetry group of a differential equation. Other important advances concern the theory of infinite dimensional Lie algebras, such as loop algebras, Kac-Moody and Virasoro algebras which frequently occur as Lie algebras of the symmetry groups of integrable equations in $2+1$ dimensions such as the Kadomtsev-Petviashvili equation. Furthermore, practical and computerizable algorithms have been proposed for finding all subgroups of a given Lie group and for recognizing Lie algebras given their structure constants.

In chapter 2 we give an introduction into group theory. Both finite and infinite groups are discussed. All the relevant concepts and definitions are introduced.

Lie groups are introduced in chapter 3. In particular, the classical Lie groups are studied in detail. The Haar measure is also discussed and examples are provided.

In chapter 4 Lie transformation groups are defined and a large number of applications are provided.

Chapter 5 is devoted to the infinitesimal transformations (vector fields) of Lie transformation groups. In particular, the three theorems of Lie are discussed.

Chapter 6 gives a comprehensive introduction into Lie algebras. We also discuss representations of Lie algebras in details. Many examples are provided to clarify the definitions and theorems. Also concepts important in theoretical physics such as Casimir operators and Cartan-Weyl basis are provided.

The form-invariance of partial differential equations under Lie transformation groups is illustrated by way of examples in chapter 7. This should be seen as an introduction to the development of the theory of invariance of differential equations by the jet bundle formalism. The Gauge transformation for the Schrödinger equation is also discussed. We also show how the electromagnetic field A_μ is coupled to the wave function ψ.

Chapter 8 deals with differential geometry. This means we consider differential forms and tensor fields. Theorems and definitions (with examples) are provided that are of importance in the application of Lie algebras to differential equations. A comprehensive introduction into differential forms and tensor fields is given.

The Lie derivative is of central importance for continuous symmetries with applications to differential equations. In chapter 9 we study invariance and conformal invariance of geometrical objects, i.e. functions, vector fields, differential forms, tensor fields, etc..

In chapter 10 the jet bundle formalism in connection with the prolongation of vector fields and (partial) differential equations is studied. The application of the Lie derivative in the jet bundle formalism is analysed to obtain the invariant Lie algebra. Explicit analytic solutions are then constructed by applying the invariant Lie algebra. These are the so-called similarity solutions which are of great theoretical and practical importance. The direct method is also introduced.

In chapter 11 the generalisation of the Lie point symmetry vector fields is considered. These generalised vector fields are known as the Lie-Bäcklund symmetry vector fields. Similarity solutions are constructed from the Lie-Bäcklund vector fields. The connection with gauge transformations is also discussed.

In chapter 12 the inverse problem is considered. This means that a partial differential equation is constructed from a given Lie algebra which is spanned by Lie point or Lie-Bäcklund symmetry vector fields.

A list of Lie symmetry vector fields of some important partial differential equations in physics is included in chapter 13. In particular the Lie symmetry vector fields for the Maxwell-Dirac equation have been calculated.

In chapter 14 the Gateaux derivative is defined. A Lie algebra is introduced using the Gateaux derivative. Furthermore, recursion operators are defined and applied. Then we can find hierarchies of integrable equations.

In chapter 15 we introduce auto-Bäcklund transformations and Bäcklund transformations.for partial and ordinary differential equations. We show that these transformations can be used to construt solutions.

For soliton equations the Lax representations are the starting point for the inverse scattering method. In chapter 16 we discuss the Lax representation. Many illustrative examples are given. Sato's theory is also included.

The important concept of conservation laws is discussed in chapter 17. The connection between conservation laws and Lie symmetry vector fields is of particular interest. Extensive use is made of the definitions and theorems of exterior differential forms. The Cartan fundamental form plays an important role regarding the Lagrange density and Hamilton density. String theory and invariants are also discussed.

In chapter 18 the Painlevé test is studied with regard to the symmetries of ordinary and partial differential equations. The Painlevé test provides an approach to study the integrability of ordinary and partial differential equations. This approach is studied and several examples are given. In particular a connection between the singularity manifold and similarity variables is presented. The connection of the Hirota technique and the Painlevé test is also discussed in detail.

Ziglin's theorem can be used to decide whether an ordinary differential equation can be integrated. This theorem is discussed in chapter 19 together we many applications.

In chapter 20 the extension of differential forms, discussed in chapter 7, to Lie algebra valued differential forms is studied. The covariant exterior derivative is defined. Then the Yang-Mills equations and self-dual Yang-