Operator Algebras and Topology

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Abstract

These notes, based on three lectures on operator algebras and topology at the "School on High Dimensional Manifold Theory" at the ICTP in Trieste, introduce a new set of tools to high dimensional manifold theory, namely techniques coming from the theory of operator algebras, in particular C^*-algebras. These are extensively studied in their own right. We will focus on the basic definitions and properties, and on their relevance to the geometry and topology of manifolds.

A central pillar of work in the theory of C^*-algebras is the Baum-Connes conjecture. This is an isomorphism conjecture, as discussed in the talks of Lück, but with a certain special flavor. Nevertheless, it has important direct applications to the topology of manifolds, it implies e.g. the Novikov conjecture. In the first chapter, the Baum-Connes conjecture will be explained and put into our context.

Another application of the Baum-Connes conjecture is to the positive scalar curvature question. This will be discussed by Stephan Stolz. It implies the so-called "stable Gromov-Lawson-Rosenberg conjecture". The unstable version of this conjecture said that, given a closed spin manifold M, a certain obstruction, living in a certain (topological) K-theory group, vanishes if and only M admits a Riemannian metric with positive scalar curvature. It turns out that this is wrong, and counterexamples will be presented in the second chapter.

The third chapter introduces another set of invariants, also using operator algebra techniques, namely L^2-cohomology, L^2-Betti numbers and other L^2-invariants. These invariants, their basic properties, and the central questions about them, are introduced in the third chapter.

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Contents

Chapter 1

Index theory and the Baum-Connes conjecture

1.1 Index theory

The Atiyah-Singer index theorem is one of the great achievements of modern mathematics. It gives a formula for the index of a differential operator (the index is by definition the dimension of the space of its solutions minus the dimension of the solution space for its adjoint operator) in terms only of topological data associated to the operator and the underlying space. There are many good treatments of this subject available, apart from the original literature (most found in [2]). Much more detailed than the present notes can be, because of constraints of length and time, are e.g. [44, 7, 32].

1.1.1 Elliptic operators and their index

We quickly review what type of operators we are looking at.

1.1.1. Definition. Let M be a smooth manifold of dimension m; E, F smooth (complex) vector bundles on M. A *differential operator* (of order d) from E to F is a $\mathbb{C}$-linear map from the space of smooth sections $C^\infty(E)$ of E to the space of smooth sections of F:

$$D\colon C^\infty(E) \to C^\infty(F),$$

such that in local coordinates and with local trivializations of the bundles it can be written in the form

$$D = \sum_{|\alpha| \leq d} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here $A_\alpha(x)$ is a matrix of smooth complex valued functions, $\alpha = (\alpha_1, \dots, \alpha_m)$ is an m-tuple of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. $\partial^{|\alpha|}/\partial x^\alpha$ is an abbreviation for $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}$. We require that $A_\alpha(x) \neq 0$ for some α with $|\alpha| = d$ (else, the operator is of order strictly smaller than d).

Let $\pi\colon T^*M \to M$ be the bundle projection of the cotangent bundle of M. We get pull-backs π^*E and π^*F of the bundles E and F, respectively, to T^*M.

The *symbol* $\sigma(D)$ of the differential operator D is the section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ on T^*M defined as follows:

In the above local coordinates, using $\xi = (\xi_1, \dots, \xi_m)$ as coordinate for the cotangent vectors in T^*M, in the fiber of (x, ξ), the symbol $\sigma(D)$ is given by multiplication with

$$\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}$.

The operator D is called *elliptic*, if $\sigma(D)_{(x,\xi)}\colon \pi^*E_{(x,\xi)} \to \pi^*F_{(x,\xi)}$ is invertible outside the zero section of T^*M, i.e. in each fiber over $(x, \xi) \in T^*M$ with $\xi \neq 0$. Observe that elliptic operators can only exist if the fiber dimensions of E and F coincide.

In other words, the symbol of an elliptic operator gives us two vector bundles over T^*M, namely π^*E and π^*F, together with a choice of an isomorphism of the fibers of these two bundles outside the zero section. If M is compact, this gives an element of the relative K-theory group $K^0(DT^*M, ST^*M)$, where DT^*M and ST^*M are the disc bundle and sphere bundle of T^*M, respectively (with respect to some arbitrary Riemannian metric).

Recall the following definition:

1.1.2. Definition. Let X be a compact topological space. We define the K-theory of X, $K^0(X)$, to be the Grothendieck group of (isomorphism classes of) complex vector bundles over X (with finite fiber dimension). More precisely, $K^0(X)$ consists of equivalence classes of pairs (E, F) of (isomorphism classes of) vector bundles over X, where $(E, F) \sim (E', F')$ if and only if there exists another vector bundle G on X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. One often writes $[E] - [F]$ for the element of $K^0(X)$ represented by (E, F).

Let Y now be a closed subspace of X. The *relative K-theory* $K^0(X, Y)$ is given by equivalence classes of triples (E, F, ϕ), where E and F are complex vector bundles over X, and $\phi\colon E|_Y \to F|_Y$ is a given isomorphism between the restrictions of E and F to Y. Then (E, F, ϕ) is isomorphic to (E', F', ϕ')

if we find isomorphisms $\alpha\colon E \to E'$ and $\beta\colon F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} E|_Y & \xrightarrow{\phi} & F|_Y \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ E'|_Y & \xrightarrow{\phi'} & F'|_Y \end{array}$$

Two pairs (E, F, ϕ) and (E', F', ϕ') are equivalent, if there is a bundle G on X such that $(E \oplus G, F \oplus G, \phi \oplus \mathrm{id})$ is isomorphic to $(E' \oplus G, F' \oplus G, \phi' \oplus \mathrm{id})$.

1.1.3. Example. The element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic differential operator D mentioned above is represented by the restriction of the bundles π^*E and π^*F to the disc bundle DT^*M, together with the isomorphism $\sigma(D)_{(x,\xi)}\colon E_{(x,\xi)} \to F_{(x,\xi)}$ for $(x, \xi) \in ST^*M$.

1.1.4. Example. Let $M = \mathbb{R}^m$ and $D = \sum_{i=1}^m (\partial/\partial_i)^2$ be the Laplace operator on functions. This is an elliptic differential operator, with symbol $\sigma(D) = \sum_{i=1}^m \xi_i^2$.

More generally, a second-order differential operator $D\colon C^\infty(E) \to C^\infty(E)$ on a Riemannian manifold M is a *generalized Laplacian*, if $\sigma(D)_{(x,\xi)} = |\xi|^2 \cdot \mathrm{id}_{E_x}$ (the norm of the cotangent vector $|\xi|$ is given by the Riemannian metric).

Notice that all generalized Laplacians are elliptic.

1.1.5. Definition. *(Adjoint operator)*
Assume that we have a differential operator $D\colon C^\infty(E) \to C^\infty(F)$ between two Hermitian bundles E and F on a Riemannian manifold (M, g). We define an L^2-inner product on $C^\infty(E)$ by the formula

$$\langle f, g\rangle_{L^2(E)} := \int_M \langle f(x), g(x)\rangle_{E_x} \, d\mu(x) \qquad \forall f, g \in C_0^\infty(E),$$

where $\langle\cdot,\cdot\rangle_{E_x}$ is the fiber-wise inner product given by the Hermitian metric, and $d\mu$ is the measure on M induced from the Riemannian metric. Here C_0^∞ is the space of smooth section with compact support. The Hilbert space completion of $C_0^\infty(E)$ with respect to this inner product is called $L^2(E)$.

The *formal adjoint* D^* of D is then defined by

$$\langle Df, g\rangle_{L^2(F)} = \langle f, D^*g\rangle_{L^2(E)} \qquad \forall f \in C_0^\infty(E),\ g \in C_0^\infty(F).$$

It turns out that exactly one operator with this property exists, which is another differential operator, and which is elliptic if and only if D is elliptic.

1.1.6. Remark. The class of differential operators is quite restricted. Many constructions one would like to carry out with differential operators automatically lead out of this class. Therefore, one often has to use *pseudodifferential operators*. Pseudodifferential operators are defined as a generalization of differential operators. There are many well written sources dealing with the theory of pseudodifferential operators. Since we will not discuss them in detail here, we omit even their precise definition and refer e.g. to [44] and [78]. What we have done so far with elliptic operators can all be extended to pseudodifferential operators. In particular, they have a symbol, and the concept of ellipticity is defined for them. When studying elliptic differential operators, pseudodifferential operators naturally appear and play a very important role. An pseudodifferential operator P (which could e.g. be a differential operator) is elliptic if and only if a pseudodifferential operator Q exists such that $PQ - \mathrm{id}$ and $QP - \mathrm{id}$ are so called *smoothing* operators, a particularly nice class of pseudodifferential operators. For many purposes, Q can be considered to act like an inverse of P, and this kind of invertibility is frequently used in the theory of elliptic operators. However, if P happens to be an elliptic differential operator of positive order, then Q necessarily is not a differential operator, but only a pseudodifferential operator.

It should be noted that almost all of the results we present here for differential operators hold also for pseudodifferential operators, and often the proof is best given using them.

We now want to state several important properties of elliptic operators.

1.1.7. Theorem. *Let M be a smooth manifold, E and F smooth finite dimensional vector bundles over M. Let $P\colon C^\infty(E) \to C^\infty(F)$ be an elliptic operator.*

Then the following holds.

(1) *Elliptic regularity:*
If $f \in L^2(E)$ is weakly in the null space of P, i.e. $\langle f, P^*g\rangle_{L^2(E)} = 0$ for all $g \in C_0^\infty(F)$, then $f \in C^\infty(E)$.

(2) *Decomposition into finite dimensional eigenspaces:*
Assume M is compact and $P = P^*$ (in particular, $E = F$). Then the set $s(P)$ of eigenvalues of P (P acting on $C^\infty(E)$) is a discrete subset of $\mathbb{R}$, each eigenspace e_λ ($\lambda \in s(P)$) is finite dimensional, and $L^2(E) = \oplus_{\lambda\in s(P)} e_\lambda$ (here we use the completed direct sum in the sense of Hilbert spaces, which means by definition that the algebraic direct sum is dense in $L^2(E)$).

(3) *If M is compact, then* $\ker(P)$ *and* $\ker(P^*)$ *are finite dimensional, and*

then we define the index of P

$$\operatorname{ind}(P) := \dim_{\mathbb{C}} \ker(P) - \dim_{\mathbb{C}} \ker(P^*).$$

(Here, we could replace $\ker(P^*)$ *by* $\operatorname{coker}(P)$*, because these two vector spaces are isomorphic).*

1.1.2 Statement of the Atiyah-Singer index theorem

There are different variants of the Atiyah-Singer index theorem. We start with a cohomological formula for the index.

1.1.8. Theorem. *Let M be a compact oriented manifold of dimension m, and $D\colon C^\infty(E) \to C^\infty(F)$ an elliptic operator with symbol $\sigma(D)$. There is a characteristic (inhomogeneous) cohomology class $\operatorname{Td}(M) \in H^*(M;\mathbb{Q})$ of the tangent bundle of M (called the complex Todd class of the complexified tangent bundle). Moreover, to the symbol is associated a certain (inhomogeneous) cohomology class $\pi_! \operatorname{ch}(\sigma(D)) \in H^*(M;\mathbb{Q})$ such that*

$$\operatorname{ind}(D) = (-1)^{m(m+1)/2} \langle \pi_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(M), [M] \rangle.$$

The class $[M] \in H_m(M;\mathbb{Q})$ is the fundamental class of the oriented manifold M, and $\langle \cdot, \cdot \rangle$ is the usual pairing between homology and cohomology.

If we start with specific operators given by the geometry, explicit calculation usually give more familiar terms on the right hand side.

For example, for the signature operator we obtain Hirzebruch's signature formula expressing the signature in terms of the L-class, for the Euler characteristic operator we obtain the Gauss-Bonnet formula expressing the Euler characteristic in terms of the Pfaffian, and for the spin or spin^c Dirac operator we obtain an $\hat{A}$-formula. For applications, these formulas prove to be particularly useful.

We give some more details about the signature operator, which we are going to use later again. To define the signature operator, fix a Riemannian metric g on M. Assume $\dim M = 4k$ is divisible by four.

The signature operator maps from a certain subspace Ω^+ of the space of differential forms to another subspace Ω^-. These subspaces are defined as follows. Define, on p-forms, the operator $\tau := i^{p(p-1)+2k} *$, where $*$ is the Hodge-$*$ operator given by the Riemannian metric, and $i^2 = -1$. Since $\dim M$ is divisible by 4, an easy calculation shows that $\tau^2 = \operatorname{id}$. We then define $\Omega^\pm$ to be the ± 1 eigenspaces of τ.

The signature operator D_{sig} is now simply defined to by $D_{sig} := d + d^*$, where d is the exterior derivative on differential forms, and $d^* = \pm * d *$ is its formal adjoint. We restrict this operator to Ω^+, and another easy calculation shows that Ω^+ is mapped to Ω^-. D_{sig} is elliptic, and a classical calculation shows that its index is the signature of M given by the intersection form in middle homology.

1.1.9. Definition. The *Hirzebruch L-class* as normalized by Atiyah and Singer is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space X a cohomology class $L(E) \in H^*(X;\mathbb{Q})$. It is characterized by the following properties:

(1) Naturality: for any map $f\colon Y \to X$ we have $L(f^*E) = f^*L(E)$.

(2) Normalization: If L is a complex line bundle with first Chern class x, then

$$L(E) = \frac{x/2}{\tanh(x/2)} = 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \in H^*(X;\mathbb{Q}).$$

(3) Multiplicativity: $L(E \oplus F) = L(E)L(F)$.

It turns out that L is a *stable* characteristic class, i.e. $L(E) = 1$ if E is a trivial bundle. This implies that L defines a map from the K-theory $K^0(X) \to H^*(X;\mathbb{Q})$.

The Atiyah-Singer index theorem now specializes to

$$\operatorname{sign}(M) = \operatorname{ind}(D_{sig}) = \langle 2^{2k}L(TM), [M]\rangle,$$

with $\dim M = 4k$ as above.

1.1.10. Remark. One direction to generalize the Atiyah-Singer index theorem is to give an index formula for manifolds with boundary. Indeed, this is achieved in the Atiyah-Patodi-Singer index theorem. However, these results are much less topological than the results for manifolds without boundary. They are not discussed in these notes.

Next, we explain the K-theoretic version of the Atiyah-Singer index theorem. It starts with the element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic operator. Given any compact manifold M, there is a well defined homomorphism

$$K^0(DT^*M, ST^*M) \to K^0(*) = \mathbb{Z},$$

constructed by embedding T^*M into high dimensional Euclidean space, then using a transfer map and Bott periodicity. The image of the symbol element under this homomorphism is denoted the *topological index* $\operatorname{ind}_t(D) \in K^0(*) = \mathbb{Z}$. The reason for the terminology is that it is obtained from the symbol only, using purely topological constructions. Now, the Atiyah-Singer index theorem states

1.1.11. Theorem. $\operatorname{ind}_t(D) = \operatorname{ind}(D)$.

1.1.3 The G-index

Let G be a finite group, or more generally a compact Lie group. The representation ring RG of G is defined to be the Grothendieck group of all finite dimensional complex representations of G, i.e. an element of RG is a formal difference $[V] - [W]$ of two finite dimensional G-representations V and W, and we have $[V] - [W] = [X] - [Y]$ if and only if $V \oplus Y \cong W \oplus X$ (strictly speaking, we have to pass to isomorphism classes of representations to avoid set theoretical problems). The direct sum of representations induces the structure of an abelian group on RG, and the tensor product makes it a commutative unital ring (the unit given by the trivial one-dimensional representation). More about this representation ring can be found e.g. in [11].

Assume now that the manifold M is a compact smooth manifold with a smooth G-action, and let E, F be complex G-vector bundles on M (this means that G acts on E and F by vector bundle automorphisms (i.e. carries fibers to fibers linearly), and the bundle projection maps are G-equivariant).

Let $D\colon C^\infty(E) \to C^\infty(F)$ be a G-equivariant elliptic differential operator.

This implies that $\ker(D)$ and $\operatorname{coker}(D)$ inherit a G-action by restriction, i.e. are finite dimensional G-representations. We define the (analytic) G-index of D to be

$$\operatorname{ind}^G(D) := [\ker(D)] - [\operatorname{coker}(D)] \in RG.$$

If G is the trivial group then $RG \cong \mathbb{Z}$ in a canonical way, and then $\operatorname{ind}^G(D)$ coincides with the usual index of D.

We can also define a *topological* equivariant index similar to the non-equivariant topological index, using transfer maps and Bott periodicity. This topological index lives in the G-equivariant K-theory of a point, which is canonically isomorphic to the representation ring RG. Again, the Atiyah-Singer index theorem says

1.1.12. Theorem. $\operatorname{ind}^G(D) = \operatorname{ind}_t^G(D) \in K_G^0(*) = RG$.

1.1.4 Families of operators and their index

Another generalization is given if we don't look at one operator on one manifold, but a family of operators on a family of manifolds. More precisely, let X be any compact topological space, $Y \to X$ a locally trivial fiber bundle with fiber M a smooth compact manifold, and structure group the diffeomorphisms of M. Let E, F be families of smooth vector bundles on Y (i.e. vector bundles which are fiber-wise smooth), and $C^\infty(E)$, $C^\infty(F)$ the continuous sections which are smooth along the fibers. Assume that $D\colon C^\infty(E) \to C^\infty(F)$ is a family $\{D_x\}$ of elliptic differential operator along the fiber $Y_x \cong M$ $(x \in X)$, i.e., in local coordinates D becomes

$$\sum_{|\alpha| \le m} A_\alpha(y,x) \frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

with $y \in M$ and $x \in X$ such that $A_\alpha(y,x)$ depends continuously on x, and each D_x is an elliptic differential operator on Y_x.

If $\dim_{\mathbb{C}} \ker(D_x)$ is independent of $x \in X$, then all of these vector spaces patch together to give a vector bundle called $\ker(D)$ on X, and similarly for the (fiber-wise) adjoint D^*. This then gives a K-theory element $[\ker(D)] - [\ker(D^*)] \in K^0(X)$.

Unfortunately, it does sometimes happen that these dimensions jump. However, using appropriate perturbations, one can always define the K-theory element

$$\operatorname{ind}(D) := [\ker(D)] - [\ker(D^*)] \in K^0(X),$$

the analytic index of the family of elliptic operators D.

There is also a family version of the construction of the topological index, giving $\operatorname{ind}_t(D) \in K^0(X)$. The Atiyah-Singer index theorem for families states:

1.1.13. Theorem. $\operatorname{ind}(D) = \operatorname{ind}_t(D) \in K^0(X)$.

The upshot of the discussion of this and the last section (for the details the reader is referred to the literature) is that the natural receptacle for the index of differential operators in various situations are appropriate K-theory groups, and much of todays index theory deals with investigating these K-theory groups.

1.2 Survey on C^*-algebras and their K-theory

More detailed references for this section are, among others, [88], [32], and [8].

1.2.1 C^*-algebras

1.2.1. Definition. A *Banach algebra* A is a complex algebra which is a complete normed space, and such that $|ab| \leq |a|\,|b|$ for each $a, b \in A$.

A $*$-*algebra* A is a complex algebra with an anti-linear involution $*\colon A \to A$ (i.e. $(\lambda a)^* = \overline{\lambda} a^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$ for all $a, b \in A$).

A *Banach* $*$-*algebra* A is a Banach algebra which is a $*$-algebra such that $|a^*| = |a|$ for all $a \in A$.

A C^*-*algebra* A is a Banach $*$-algebra which satisfies $|a^* a| = |a|^2$ for all $a \in A$.

Alternatively, a C^*-algebra is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed subalgebra of the algebra of bounded operators on some Hilbert space H (this is the Gelfand-Naimark representation theorem, compare e.g. [32, 1.6.2]).

A C^*-algebra A is called separable if there exists a countable dense subset of A.

1.2.2. Example. If X is a compact topological space, then $C(X)$, the algebra of complex valued continuous functions on X, is a commutative C^*-algebra (with unit). The adjoint is given by complex conjugation: $f^*(x) = \overline{f(x)}$, the norm is the supremum-norm.

Conversely, it is a theorem that every abelian unital C^*-algebra is isomorphic to $C(X)$ for a suitable compact topological space X [32, Theorem 1.3.12].

Assume X is locally compact, and set

$$C_0(X) := \{f\colon X \to \mathbb{C} \mid f \text{ continuous}, f(x) \xrightarrow{x\to\infty} 0\}.$$

Here, we say $f(x) \to 0$ for $x \to \infty$, or$f$ *vanishes at infinity*, if for all $\epsilon > 0$ there is a compact subset K of X with $|f(x)| < \epsilon$ whenever $x \in X - K$. This is again a commutative C^*-algebra (we use the supremum norm on $C_0(X)$), and it is unital if and only if X is compact (in this case, $C_0(X) = C(X)$).

1.2.2 K_0 of a ring

Suppose R is an arbitrary ring with 1 (not necessarily commutative). A module M over R is called finitely generated projective, if there is another R-module N and a number $n \geq 0$ such that

$$M \oplus N \cong R^n.$$

This is equivalent to the assertion that the matrix ring $M_n(R) = End_R(R^n)$ contains an idempotent e, i.e. with $e^2 = e$, such that M is isomorphic to the image of e, i.e. $M \cong eR^n$.

1.2.3. Example. Description of projective modules.

(1) If R is a field, the finitely generated projective R-modules are exactly the finite dimensional vector spaces. (In this case, every module is projective).

(2) If $R = \mathbb{Z}$, the finitely generated projective modules are the free abelian groups of finite rank

(3) Assume X is a compact topological space and $A = C(X)$. Then, by the Swan-Serre theorem [84], M is a finitely generated projective A-module if and only if M is isomorphic to the space $\Gamma(E)$ of continuous sections of some complex vector bundle E over X.

1.2.4. Definition. Let R be any ring with unit. $K_0(R)$ is defined to be the Grothendieck group of finitely generated projective modules over R, i.e. the group of equivalence classes $[(M,N)]$ of pairs of (isomorphism classes of) finitely generated projective R-modules M, N, where $(M,N) \equiv (M',N')$ if and only if there is an $n \geq 0$ with

$$M \oplus N' \oplus R^n \cong M' \oplus N \oplus R^n.$$

The group composition is given by

$$[(M,N)] + [(M',N')] := [(M \oplus M', N \oplus N')].$$

We can think of (M,N) as the formal difference of modules $M - N$.

Any unital ring homomorphism $f\colon R \to S$ induces a map

$$f_*\colon K_0(R) \to K_0(S)\colon [M] \mapsto [S \otimes_R M],$$

where S becomes a right R-module via f. We obtain that K_0 is a covariant functor from the category of unital rings to the category of abelian groups.

1.2.5. Example. Calculation of K_0.

- If R is a field, then $K_0(R) \cong \mathbb{Z}$, the isomorphism given by the dimension: $\dim_R(M,N) := \dim_R(M) - \dim_R(N)$.
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$, given by the rank.
- If X is a compact topological space, then $K_0(C(X)) \cong K^0(X)$, the topological K-theory given in terms of complex vector bundles. To each vector bundle E one associates the $C(X)$-module $\Gamma(E)$ of continuous sections of E.

- Let G be a discrete group. The group algebra $\mathbb{C}G$ is a vector space with basis G, and with multiplication coming from the group structure, i.e. given by $g \cdot h = (gh)$.

  If G is a finite group, then $K_0(\mathbb{C}G)$ is the complex representation ring of G.

1.2.3 K-Theory of C^*-algebras

1.2.6. Definition. Let A be a unital C^*-algebra. Then $K_0(A)$ is defined as in Definition 1.2.4, i.e. by forgetting the topology of A.

1.2.3.1 K-theory for non-unital C^*-algebras

When studying (the K-theory of) C^*-algebras, one has to understand morphisms $f\colon A \to B$. This necessarily involves studying the kernel of f, which is a closed ideal of A, and hence a *non-unital* C^*-algebra. Therefore, we proceed by defining the K-theory of C^*-algebras without unit.

1.2.7. Definition. To any C^*-algebra A, with or without unit, we assign in a functorial way a new, unital C^*-algebra A_+ as follows. As $\mathbb{C}$-vector space, $A_+ := A \oplus \mathbb{C}$, with product

$$(a,\lambda)(b,\mu) := (ab + \lambda a + \mu b, \lambda\mu) \qquad \text{for } (a,\lambda),(b,\mu) \in A \oplus \mathbb{C}.$$

The unit is given by $(0,1)$. The star-operation is defined as $(a,\lambda)^* := (a^*, \overline{\lambda})$, and the new norm is given by

$$|(a,\lambda)| = \sup\{|ax + \lambda x| \mid x \in A \text{ with } |x| = 1\}$$

1.2.8. Remark. A is a closed ideal of A_+, the kernel of the canonical projection $A_+ \twoheadrightarrow \mathbb{C}$ onto the second factor. If A itself is unital, the unit of A is of course different from the unit of A_+.

1.2.9. Example. Assume X is a locally compact space, and let $X_+ := X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X)_+ \cong C(X_+).$$

The ideal $C_0(X)$ of $C_0(X)_+$ is identified with the ideal of those functions $f \in C(X_+)$ such that $f(\infty) = 0$.

1.2.10. Definition. For an arbitrary C^*-algebra A (not necessarily unital) define

$$K_0(A) := \ker(K_0(A_+) \to K_0(\mathbb{C})).$$

Any C^*-algebra homomorphisms $f\colon A \to B$ (not necessarily unital) induces a unital homomorphism $f_+\colon A_+ \to B_+$. The induced map

$$(f_+)_*\colon K_0(A_+) \to K_0(B_+)$$

maps the kernel of the map $K_0(A_+) \to K_0(\mathbb{C})$ to the kernel of $K_0(B_+) \to K_0(\mathbb{C})$. This means it restricts to a map $f_*\colon K_0(A) \to K_0(B)$. We obtain a covariant functor from the category of (not necessarily unital) C^*-algebras to abelian groups.

Of course, we need the following result.

1.2.11. Proposition. *If A is a unital C^*-algebra, the new and the old definition of $K_0(A)$ are canonically isomorphic.*

1.2.3.2 Higher topological K-groups

We also want to define higher topological K-theory groups. We have an ad hoc definition using suspensions (this is similar to the corresponding idea in topological K-theory of spaces). For this we need the following.

1.2.12. Definition. Let A be a C^*-algebra. We define the cone CA and the suspension SA as follows.

$$CA := \{f\colon [0,1] \to A \mid f(0) = 0\}$$
$$SA := \{f\colon [0,1] \to A \mid f(0) = 0 = f(1)\}.$$

These are again C^*-algebras, using pointwise operations and the supremum norm.

Inductively, we define

$$S^0A := A \qquad S^nA := S(S^{n-1}A) \quad \text{for } n \geq 1.$$

1.2.13. Definition. Assume A is a C^*-algebra. For $n \geq 0$, define

$$K_n(A) := K_0(S^nA).$$

These are the *topological K-theory groups of A*. For each $n \geq 0$, we obtain a functor from the category of C^*-algebras to the category of abelian groups.

For unital C^*-algebras, we can also give a more direct definition of higher K-groups (in particular useful for K_1, which is then defined in terms of (classes of) invertible matrices). This is done as follows:

1.2.14. Definition. Let A be a unital C^*-algebra. Then $Gl_n(A)$ becomes a topological group, and we have continuous embeddings

$$Gl_n(A) \hookrightarrow Gl_{n+1}(A)\colon X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We set $Gl_\infty(A) := \lim_{n\to\infty} Gl_n(A)$, and we equip $Gl_\infty(A)$ with the direct limit topology.

1.2.15. Proposition. *Let A be a unital C^*-algebra. If $k \geq 1$, then*

$$K_k(A) = \pi_{k-1}(Gl_\infty(A))(\cong \pi_k(BGl_\infty(A))).$$

Observe that any unital morphism $f\colon A \to B$ of unital C^*-algebras induces a map $Gl_n(A) \to Gl_n(B)$ and therefore also between $\pi_k(Gl_\infty(A))$ and $\pi_k(Gl_\infty(B))$. This map coincides with the previously defined induced map in topological K-theory.

1.2.16. Remark. Note that the topology of the C^*-algebra enters the definition of the higher topological K-theory of A, and in general the topological K-theory of A will be vastly different from the algebraic K-theory of the algebra underlying A. For connections in special cases, compare [83].

1.2.17. Example. It is well known that $Gl_n(\mathbb{C})$ is connected for each $n \in \mathbb{N}$. Therefore

$$K_1(\mathbb{C}) = \pi_0(Gl_\infty(\mathbb{C})) = 0.$$

A very important result about K-theory of C^*-algebras is the following long exact sequence. A proof can be found e.g. in [32, Proposition 4.5.9].

1.2.18. Theorem. *Assume I is a closed ideal of a C^*-algebra A. Then, we get a short exact sequence of C^*-algebras $0 \to I \to A \to A/I \to 0$, which induces a long exact sequence in K-theory*

$$\to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(I) \to \cdots \to K_0(A/I).$$

1.2.4 Bott periodicity and the cyclic exact sequence

One of the most important and remarkable results about the K-theory of C^*-algebras is Bott periodicity, which can be stated as follows.

1.2.19. Theorem. *Assume A is a C^*-algebra. There is a natural isomorphism, called the Bott map*

$$K_0(A) \to K_0(S^2 A),$$

which implies immediately that there are natural isomorphism

$$K_n(A) \cong K_{n+2}(A) \qquad \forall n \geq 0.$$

1.2.20. Remark. Bott periodicity allows us to define $K_n(A)$ for each $n \in \mathbb{Z}$, or to regard the K-theory of C^*-algebras as a $\mathbb{Z}/2$-graded theory, i.e. to talk of $K_n(A)$ with $n \in \mathbb{Z}/2$. This way, the long exact sequence of Theorem 1.2.18 becomes a (six-term) cyclic exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow{\scriptstyle \mu_*} \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

The connecting homomorphism μ_* is the composition of the Bott periodicity isomorphism and the connecting homomorphism of Theorem 1.2.18.

1.2.5 The C^*-algebra of a group

Let Γ be a discrete group. Define $l^2(\Gamma)$ to be the Hilbert space of square summable complex valued functions on Γ. We can write an element $f \in l^2(\Gamma)$ as a sum $\sum_{g\in\Gamma} \lambda_g g$ with $\lambda_g \in \mathbb{C}$ and $\sum_{g\in\Gamma} |\lambda_g|^2 < \infty$.

We defined the *complex group algebra* (often also called the *complex group ring*) $\mathbb{C}\Gamma$ to be the complex vector space with basis the elements of Γ (this can also be considered as the space of complex valued functions on Γ with finite support, and as such is a subspace of $l^2(\Gamma)$). The product in $\mathbb{C}\Gamma$ is induced by the multiplication in Γ, namely, if $f = \sum_{g\in\Gamma} \lambda_g g, u = \sum_{g\in\Gamma} \mu_g g \in \mathbb{C}\Gamma$, then

$$(\sum_{g\in\Gamma} \lambda_g g)(\sum_{g\in\Gamma} \mu_g g) := \sum_{g,h\in\Gamma} \lambda_g \mu_h (gh) = \sum_{g\in\Gamma} \left(\sum_{h\in\Gamma} \lambda_h \mu_{h^{-1}g} \right) g.$$

This is a convolution product.

We have the *left regular representation* λ_Γ of Γ on $l^2(\Gamma)$, given by

$$\lambda_\Gamma(g) \cdot (\sum_{h\in\Gamma} \lambda_h h) := \sum_{h\in\Gamma} \lambda_h gh$$

for $g \in \Gamma$ and $\sum_{h\in\Gamma} \lambda_h h \in l^2(\Gamma)$.

This unitary representation extends linearly to $\mathbb{C}\Gamma$.

The *reduced C^*-algebra* $C_r^*\Gamma$ of Γ is defined to be the norm closure of the image $\lambda_\Gamma(\mathbb{C}\Gamma)$ in the C^*-algebra of bounded operators on $l^2(\Gamma)$.

1.2.21. Remark. It's no surprise that there is also a *maximal C^*-algebra* $C^*_{max}\Gamma$ of a group Γ. It is defined using not only the left regular representation of Γ, but simultaneously all of its representations. We will not make use of $C^*_{max}\Gamma$ in these notes, and therefore will not define it here.

Given a topological group G, one can define C^*-algebras C_r^*G and $C^*_{max}G$ which take the topology of G into account. They actually play an important role in the study of the Baum-Connes conjecture, which can be defined for (almost arbitrary) topological groups, but again we will not cover this subject here. Instead, we will throughout stick to discrete groups.

1.2.22. Example. If Γ is finite, then $C_r^*\Gamma = \mathbb{C}\Gamma$ is the complex group ring of Γ.

In particular, in this case $K_0(C_r^*\Gamma) \cong R(\Gamma)$ coincides with the (additive group of) the complex representation ring of Γ.

1.3 The Baum-Connes conjecture

The Baum-Connes conjecture relates an object from algebraic topology, namely the K-homology of the classifying space of a given group Γ, to representation theory and the world of C^*-algebras, namely to the K-theory of the reduced C^*-algebra of Γ.

Unfortunately, the material is very technical. Because of lack of space and time we can not go into the details (even of some of the definitions). We recommend the sources [86], [87], [32], [4], [58] and [8].

1.3.1 The Baum-Connes conjecture for torsion-free groups

1.3.1. Definition. Let X be any CW-complex. $K_*(X)$ is the K-homology of X, where K-homology is the homology theory dual to topological K-theory. If BU is the spectrum of topological K-theory, and X_+ is X with a disjoint basepoint added, then

$$K_n(X) := \pi_n(X_+ \wedge BU).$$

1.3.2. Definition. Let Γ be a discrete group. A classifying space $B\Gamma$ for Γ is a CW-complex with the property that $\pi_1(B\Gamma) \cong \Gamma$, and $\pi_k(B\Gamma) = 0$ if $k \neq 1$. A classifying space always exists, and is unique up to homotopy equivalence. Its universal covering $E\Gamma$ is a contractible CW-complex with a free cellular Γ-action, the so called *universal space for* Γ*-actions.*

1.3.3. Remark. In the literature about the Baum-Connes conjecture, one will often find the definition

$$RK_n(X) := \varinjlim K_n(Y),$$

where the limit is taken over all finite subcomplexes Y of X. Note, however, that K-homology (like any homology theory in algebraic topology) is compatible with direct limits, which implies $RK_n(X) = K_n(X)$ as defined above. The confusion comes from the fact that operator algebraists often use Kasparov's bivariant KK-theory to define $K_*(X)$, and this coincides with the homotopy theoretic definition only if X is compact.

Recall that a group Γ is called torsion-free, if $g^n = 1$ for $g \in \Gamma$ and $n > 0$ implies that $g = 1$.

We can now formulate the Baum-Connes conjecture for torsion-free discrete groups.

1.3.4. Conjecture. *Assume* Γ *is a torsion-free discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\overline{\mu}_*\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma) \tag{1.3.5}$$

(which will be defined later). The Baum-Connes conjecture *says that this map is an isomorphism.*

1.3.6. Example. The map $\overline{\mu}_*$ of Equation (1.3.5) is also defined if Γ is not torsion-free. However, in this situation it will in general not be an isomorphism. This can already be seen if $\Gamma = \mathbb{Z}/2$. Then $C_r^*\Gamma = \mathbb{C}\Gamma \cong \mathbb{C}\oplus\mathbb{C}$ as a $\mathbb{C}$-algebra. Consequently,

$$K_0(C_r^*\Gamma) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{1.3.7}$$

On the other hand, using the homological Chern character,

$$K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{n=0}^{\infty} H_{2n}(B\Gamma; \mathbb{Q}) \cong \mathbb{Q}. \tag{1.3.8}$$

(Here we use the fact that the rational homology of every finite group is zero in positive degrees, which follows from the fact that the transfer homomorphism $H_k(B\Gamma; \mathbb{Q}) \to H_k(\{1\}; \mathbb{Q})$ is (with rational coefficients) up to a factor